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A Proposed Framework for Fuzzy and Neutrosophic Extensions of Service Integration and Management (SIAM) and ITIL-Based Models

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ABSTRACT

This paper surveys several uncertainty-handling paradigms—fuzzy sets, intuitionistic fuzzy sets, and neutrosophic sets—and demonstrates how they can enrich IT Service Management (ITSM). We then introduce two new frameworks—Fuzzy Service Integration and Management (FSIM) and Neutrosophic Service Integration and Management (NSIM)—which embed these uncertainty models into Service Integration and Management (SIAM). FSIM uses fuzzy membership functions to capture imprecision in quality, cost, and coordination overhead, while NSIM extends this to neutrosophic triples that also quantify indeterminacy and contradiction. Finally, we explore how ITIL best practices can be fused with fuzzy and neutrosophic logic to create more adaptive, resilient service-management processes.

1. Preliminaries

This section provides an introduction to the foundational concepts and definitions required for the discussions in this paper. In addition, all concepts addressed herein are assumed to be finite rather than infinite.

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1.1 Fuzzy Set and Neutrosophic Set

1.1.1 Fuzzy Set

A fuzzy set assigns to each element a degree of membership in the interval $[0,1]$, thus enabling the representation of uncertainty through a continuum of partial membership values, rather than relying on strict binary classification [(1), (2)]. Related concepts include the Intuitionistic Fuzzy Set [(3), (4)], Picture Fuzzy Set [(5)–(7)], Hesitant Fuzzy Set [(8)–(10)], and Bipolar Fuzzy Set [(11), (12)], all of which provide nuanced generalizations for handling different types of uncertainty.

We present below the relevant definitions, including those of these extended frameworks.

Definition 1.1 (Universal Set). *A universal set, denoted by U, is the set that contains all elements under consideration in a particular context. Every set discussed is assumed to be a subset of U.*

Definition 1.2 (Fuzzy Set). *[(1), (13)] A Fuzzy set τ in a non-empty universe Y is a mapping $\tau : Y \to [0,1]$. A fuzzy relation on Y is a fuzzy subset δ in $Y \times Y$. If τ is a fuzzy set in Y and δ is a fuzzy relation on Y, then δ is called a fuzzy relation on τ if*

$$\delta(y,z) \leq \min\{\tau(y), \tau(z)\} \quad \text{for all } y,z \in Y.$$

Example 1.3 (Fuzzy Set for Employee Performance in Management). *Let the universe of discourse be*

$$Y = \{\textit{Ayano}, \textit{Shinya}, \textit{Yutaka}, \textit{Yasuha}\},$$

and define a fuzzy set τ on Y that represents "high performance" as judged by management. Specify $\tau\colon Y \to [0,1]$ by

$$\tau(\textit{Ayano}) = 0.95,$$
$$\tau(\textit{Shinya}) = 0.75,$$
$$\tau(\textit{Yutaka}) = 0.60,$$
$$\tau(\textit{Yasuha}) = 0.30.$$

Here:

- $\tau(\textit{Ayano}) = 0.95$ *indicates that Ayano is almost fully regarded as a high performer.*
- $\tau(\textit{Shinya}) = 0.75$ *reflects that Shinya is largely considered high-performing, but with some reservations.*
- $\tau(\textit{Yutaka}) = 0.60$ *shows moderate performance in management's view.*
- $\tau(\textit{Yasuha}) = 0.30$ *means Yasuha is only weakly classified as a high performer.*

If we also define a fuzzy relation δ on Y to capture "peer endorsement," we require for all $y,z \in Y$:

$$\delta(y,z) \ \leq \ \min\{\tau(y), \tau(z)\}.$$

For instance, if Ayano and Shinya endorse each other strongly, we might set

$$\delta(\textit{Ayano}, \textit{Shinya}) \ = \ 0.70 \ \leq \ \min\{0.95, 0.75\} = 0.75.$$

This Fuzzy Set model allows management to rank, compare, and make decisions that incorporate gradual judgments of employee performance.

1.1.2 Neutrosophic Set

Neutrosophic Sets extend Fuzzy Sets by incorporating the concept of indeterminacy, thereby addressing situations that are neither entirely true nor entirely false. This framework provides a more flexible representation of uncertainty and ambiguity [(14)–(18)]. Their definitions are presented below.

Definition 1.4 (Neutrosophic Set)**.** *[(14), (19)] Let X be a non-empty set. A Neutrosophic Set (NS) A on X is characterized by three membership functions:*

$$T_A : X \to [0,1], \quad I_A : X \to [0,1], \quad F_A : X \to [0,1],$$

where for each $x \in X$, the values $T_A(x)$, $I_A(x)$, and $F_A(x)$ represent the degrees of truth, indeterminacy, and falsity, respectively. These values satisfy the following condition:

$$0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

Example 1.5 (Neutrosophic Set for Supplier Reliability in Management)**.** *Consider the universe of discourse*

$$X = \{\text{Supplier A}, \ \text{Supplier B}, \ \text{Supplier C}\},$$

and let R be a Neutrosophic Set on X that models the reliability of each supplier from a management perspective. Define the three membership functions T_R, I_R, and F_R as follows:

$$\begin{aligned} &T_R(\text{Supplier A}) = 0.85, \quad I_R(\text{Supplier A}) = 0.10, \quad F_R(\text{Supplier A}) = 0.05, \\ &T_R(\text{Supplier B}) = 0.60, \quad I_R(\text{Supplier B}) = 0.25, \quad F_R(\text{Supplier B}) = 0.15, \\ &T_R(\text{Supplier C}) = 0.40, \quad I_R(\text{Supplier C}) = 0.30, \quad F_R(\text{Supplier C}) = 0.30. \end{aligned}$$

Here:

- *$T_R(x)$ measures the degree to which management believes x is reliable (truth),*
- *$I_R(x)$ captures the uncertainty or lack of information about x (indeterminacy),*
- *$F_R(x)$ quantifies the degree to which x is considered unreliable (falsity).*

For example, Supplier A is judged highly reliable ($T_R = 0.85$) with low uncertainty ($I_R = 0.10$) and minimal unreliability ($F_R = 0.05$). In contrast, Supplier C has moderate reliability ($T_R = 0.40$), significant uncertainty ($I_R = 0.30$), and equally significant doubts ($F_R = 0.30$). This Neutrosophic representation allows management to make procurement decisions that explicitly account for both confidence and ambiguity in supplier assessments.

2. Main Results of This Paper

The results of this paper are presented as follows.

2.1 Mathematical Framework of Service Integration and Management

We present a formal specification of the SIAM paradigm, in which multiple vendors are coordinated to deliver integrated IT services to business users [(20)–(23)]. Classical ITSM frameworks have also been studied extensively [(24)–(26)].

Definition 2.1 (Mathematical Framework for SIAM)**.** *Let*

- $S = \{s_1, \ldots, s_n\}$ *be the finite set of services;*
- $P = \{p_1, \ldots, p_m\}$ *be the finite set of providers;*
- $M\colon S \to \mathcal{P}(P) \setminus \{\emptyset\}$ *assign to each* $s \in S$ *the nonempty subset* $M(s) \subseteq P$ *of providers delivering* s*;*
- *for each* $s \in S$ *and* $p \in M(s)$*, let*
  - $q(p, s) \in [0, 1]$ *be the quality score of provider* p *for service* s*, and*
  - $c(p, s) \in \mathbb{R}^+$ *be the cost charged by* p *for* s*;*
- $\varphi\colon \mathcal{P}([0, 1]) \to [0, 1]$ *be an aggregation operator (e.g. the arithmetic mean) that fuses individual quality scores into a single composite rating;*
- $\xi\colon S \to \mathbb{R}^+$ *denote the integrator's coordination overhead.*

Then the SIAM model is defined by the tuple

$$\mathcal{SIAM} = \big(S,\ P,\ M,\ \{q(p, s)\},\ \{c(p, s)\},\ \varphi,\ \xi\big).$$

For each $s \in S$*, the overall service quality and total cost are given by*

$$Q(s) = \varphi\big(\{\, q(p, s) : p \in M(s)\}\big), \qquad C(s) = \sum_{p \in M(s)} c(p, s) + \xi(s).$$

Remark 2.2. *Traditional ITSM centralizes service delivery within a single organization, whereas SIAM explicitly maps each service to multiple providers via* M*, merges their quality metrics through* φ*, and incorporates the integrator's coordination overhead* ξ *into the cost.*

Example 2.3 (Cloud Hosting under SIAM)**.** *A cloud hosting service offers scalable compute and storage on demand [(27)–(29)]. Suppose service* s_1 *is delivered by providers* p_1 *and* p_2*, so*

$$M(s_1) = \{p_1, p_2\}.$$

If

$$q(p_1, s_1) = 0.9, \quad q(p_2, s_1) = 0.8,$$

then using the arithmetic mean,

$$Q(s_1) = \frac{0.9 + 0.8}{2} = 0.85.$$

Likewise, if

$$c(p_1, s_1) = 200, \quad c(p_2, s_1) = 250, \quad \xi(s_1) = 50,$$

it follows that

$$C(s_1) = 200 + 250 + 50 = 500.$$

Example 2.4 (Enterprise Email Service via SIAM)**.** *An organization requires a reliable, business-critical email service. It engages two providers:*

$$S = \{\, s_{\textit{email}} \,\}, \quad P = \{\textit{ExchangeCo},\ \textit{GMailCorp}\}.$$

The mapping is

$$M(s_{\textit{email}}) = \{\textit{ExchangeCo},\ \textit{GMailCorp}\}.$$

Suppose the measured quality scores and costs are:

$$q(\textit{ExchangeCo}, s_{\textit{email}}) = 0.95, \quad c(\textit{ExchangeCo}, s_{\textit{email}}) = \$1\,000;$$

$$q(\textit{GMailCorp}, s_{\textit{email}}) = 0.90, \quad c(\textit{GMailCorp}, s_{\textit{email}}) = \$1\,200.$$

The integrator's coordination overhead is $\xi(s_{\textit{email}}) = \$250$. *Using the arithmetic mean for* φ*,*

$$Q(s_{\textit{email}}) = \frac{0.95 + 0.90}{2} = 0.925,$$

$$C(s_{\textit{email}}) = 1\,000 + 1\,200 + 250 = \$2\,450.$$

Thus, the enterprise email service achieves overall quality 0.925 at a total operational cost of $2,450 under the SIAM model.

Theorem 2.5 (Bounded Overall Quality)**.** *Let* $\varphi\colon \mathcal{P}([0,1]) \to [0,1]$ *satisfy*

$$\min X \;\le\; \varphi(X) \;\le\; \max X \quad \text{for every nonempty } X \subseteq [0,1].$$

Then for each service $s \in S$*,*

$$\min_{p \in M(s)} q(p,s) \;\le\; Q(s) \;\le\; \max_{p \in M(s)} q(p,s).$$

Proof. Define the set of quality scores for service s by

$$X_s \;=\; \{\, q(p,s) \mid p \in M(s)\}.$$

Since $M(s) \neq \emptyset$, X_s is nonempty and $X_s \subseteq [0,1]$. By the model definition,

$$Q(s) = \varphi\big(X_s\big).$$

Applying the bounding property of φ to X_s yields

$$\min X_s \;\le\; \varphi(X_s) \;\le\; \max X_s,$$

i.e.

$$\min_{p \in M(s)} q(p,s) \;\le\; Q(s) \;\le\; \max_{p \in M(s)} q(p,s),$$

as claimed. □

Theorem 2.6 (Monotonicity of Overall Quality)**.** *Assume* φ *is non-decreasing in each argument: if* $(x_1, \ldots, x_n)$ *and* $(x'_1, \ldots, x'_n)$ *satisfy* $x_i \le x'_i$ *for all* i*, then*

$$\varphi(\{x_1, \ldots, x_n\}) \;\le\; \varphi(\{x'_1, \ldots, x'_n\}).$$

Fix a service s *and suppose one provider* $p \in M(s)$ *improves its quality from* $q(p,s)$ *to* $q'(p,s) > q(p,s)$*, with all other* $q(r,s)$ *unchanged. Denote*

$$X = \{q(r,s) \mid r \in M(s)\}, \quad X' = \big(X \setminus \{q(p,s)\}\big) \cup \{q'(p,s)\}.$$

Then $X \le X'$ *coordinatewise, so by monotonicity of* φ*,*

$$Q(s) = \varphi(X) \;\le\; \varphi(X') = Q'(s),$$

showing that $Q(s)$ *cannot decrease when any single provider's quality increases.*

Proof. Let us enumerate $M(s) = \{p_1, \ldots, p_k\}$ so that $p_1 = p$ is the provider whose score increases. Define

$$x_i = q(p_i, s), \quad x'_1 = q'(p, s), \quad x'_i = x_i \quad (i > 1).$$

Then $x_i \le x'_i$ for all i, and

$$X = \{x_1, \ldots, x_k\}, \quad X' = \{x'_1, \ldots, x'_k\}.$$

By assumption on φ,

$$\varphi(X) \le \varphi(X').$$

By definition, $\varphi(X) = Q(s)$ and $\varphi(X') = Q'(s)$, so $Q(s) \le Q'(s)$, as required. □

Theorem 2.7 (Singleton Reduction). *If for some service $s \in S$ the integrator uses exactly one provider, $M(s) = \{p\}$, then*

$$Q(s) = q(p, s), \qquad C(s) = c(p, s) + \xi(s).$$

Proof. When $M(s) = \{p\}$, the set of quality scores is $X_s = \{q(p, s)\}$. Any aggregation operator φ that operates on a singleton must return that singleton value, i.e.

$$Q(s) = \varphi\big(\{q(p, s)\}\big) = q(p, s).$$

Similarly, by the cost definition,

$$C(s) = \sum_{r \in M(s)} c(r, s) + \xi(s) = c(p, s) + \xi(s).$$

This completes the proof. □

Theorem 2.8 (Strict Monotonicity of Total Cost). *For each service $s \in S$, the total cost*

$$C(s) = \sum_{p \in M(s)} c(p, s) \; + \; \xi(s)$$

is strictly increasing in each individual cost $c(p, s)$ and in the overhead $\xi(s)$.

Proof. View $C(s)$ as a function of the variables $\{c(p, s) \mid p \in M(s)\}$ and $\xi(s)$:

$$C(s; c_1, \ldots, c_k, \xi) = c_1 + \cdots + c_k + \xi.$$

If any one cost c_i increases by a positive increment $\delta > 0$, then

$$C(s; c_1, \ldots, c_i + \delta, \ldots, c_k, \xi) = \big(c_1 + \cdots + c_i + \cdots + c_k + \xi\big) + \delta = C(s) + \delta > C(s).$$

Similarly, increasing ξ by any $\epsilon > 0$ raises $C(s)$ by ϵ. Hence $C(s)$ is strictly increasing in each of its arguments. □

2.2 Fuzzy Service Integration and Management

Fuzzy Service Integration and Management extends SIAM by modeling service quality, cost, and overhead as fuzzy values to manage uncertainty. The definitions and related concepts of Fuzzy Service Integration and Management are provided below.

Definition 2.9 (Mathematical Framework for Fuzzy Service Integration and Management)**.** *To extend the above framework to capture uncertainty, we define the* Fuzzy Service Integration and Management *(FSIM) system as follows. In this extension:*

- *The set S of services, the set P of providers, the mapping M, and the overhead cost function ξ remain as in the conventional framework.*
- *For each service $s \in S$ and each provider $p \in M(s)$, we now assign a* fuzzy quality $\tilde{q}(p,s)$ *and a* fuzzy cost $\tilde{c}(p,s)$, *each represented as a fuzzy number (for example, a triangular fuzzy number) that belongs to the appropriate domain (i.e., $\tilde{q}(p,s) \subseteq [0,1]$ and $\tilde{c}(p,s) \subseteq \mathbb{R}^+$).*
- *Let $\tilde{\varphi}$ be a fuzzy aggregation operator that combines a set of fuzzy quality values into a single fuzzy quality value.*

Then, the FSIM system is modeled by the tuple

$$\mathcal{FSIM} = \Big(S,\, P,\, M,\, \{\tilde{q}(p,s)\}_{p\in M(s),\, s\in S},\, \{\tilde{c}(p,s)\}_{p\in M(s),\, s\in S},\, \tilde{\varphi},\, \tilde{\xi}\Big),$$

where $\tilde{\xi} : S \to \tilde{P}(\mathbb{R}^+)$ is a fuzzy integration overhead cost function. For each service $s \in S$, the overall fuzzy quality and fuzzy cost are defined as:

$$\tilde{Q}(s) = \tilde{\varphi}\Big(\{\tilde{q}(p,s) : p \in M(s)\}\Big),$$

$$\tilde{C}(s) = \bigoplus_{p\in M(s)} \tilde{c}(p,s) \oplus \tilde{\xi}(s),$$

where $\oplus$ denotes the fuzzy addition operation.

Example 2.10 (Cloud Hosting Service via Fuzzy Service Integration and Management)**.** *Consider a cloud hosting service s_1 delivered by two providers p_1 and p_2; hence,*

$$M(s_1) = \{p_1, p_2\}.$$

Assume the following fuzzy evaluations:

Fuzzy Quality:

- *$\tilde{q}(p_1, s_1)$ is represented by the triangular fuzzy number $(0.85,\, 0.90,\, 0.95)$.*
- *$\tilde{q}(p_2, s_1)$ is represented by the triangular fuzzy number $(0.80,\, 0.85,\, 0.90)$.*

Using a simple aggregation operator (for instance, the arithmetic mean of the modal values), the overall fuzzy quality for s_1 is approximated by

$$\tilde{Q}(s_1) \approx \left(\frac{0.85+0.80}{2},\, \frac{0.90+0.85}{2},\, \frac{0.95+0.90}{2}\right) = (0.825,\, 0.875,\, 0.925).$$

Fuzzy Cost: *Suppose:*

- *$\tilde{c}(p_1, s_1)$ is given by the triangular fuzzy number $(180,\, 200,\, 220)$ dollars.*
- *$\tilde{c}(p_2, s_1)$ is given by the triangular fuzzy number $(190,\, 210,\, 230)$ dollars.*

Assume the fuzzy integration overhead for the service is

$$\tilde{\xi}(s_1) = (30,\ 40,\ 50)\ \textit{dollars}.$$

The overall fuzzy cost is obtained via fuzzy addition:

$$\tilde{C}(s_1) = \tilde{c}(p_1, s_1) \oplus \tilde{c}(p_2, s_1) \oplus \tilde{\xi}(s_1).$$

Using fuzzy addition on triangular fuzzy numbers (which is defined componentwise), we have

$$\tilde{C}(s_1) = (180 + 190 + 30,\ 200 + 210 + 40,\ 220 + 230 + 50) = (400,\ 450,\ 500)\ \textit{dollars}.$$

Thus, the cloud hosting service provided by the two vendors has an overall fuzzy quality of approximately $(0.825,\ 0.875,\ 0.925)$ *and an overall fuzzy cost of* $(400,\ 450,\ 500)$ *dollars.*

Example 2.11 (Online Banking Service via FSIM). *An online banking service provides secure electronic access to financial accounts, letting users manage balances, make payments, and transact remotely (cf.[(30), (31)]). Consider an online banking service, denoted by* s_{bank}*, which is provided by two providers* p_1 *and* p_2*:*

$$M(s_{\mathit{bank}}) = \{p_1, p_2\}.$$

Assume the following fuzzy evaluations:

Fuzzy Quality:

- $\tilde{q}(p_1, s_{\mathit{bank}})$ *is expressed as the triangular fuzzy number* $(0.90,\ 0.92,\ 0.94)$.
- $\tilde{q}(p_2, s_{\mathit{bank}})$ *is expressed as* $(0.85,\ 0.88,\ 0.91)$.

Using a fuzzy aggregation operator (e.g., the arithmetic mean of the modal values), the overall fuzzy quality for s_{bank} *is approximated by:*

$$\tilde{Q}(s_{\mathit{bank}}) \approx \left(\frac{0.90 + 0.85}{2}, \frac{0.92 + 0.88}{2}, \frac{0.94 + 0.91}{2}\right) = (0.875,\ 0.90,\ 0.925).$$

Fuzzy Cost:

- $\tilde{c}(p_1, s_{\mathit{bank}})$ *is given by the triangular fuzzy number* $(300,\ 320,\ 340)$ *dollars.*
- $\tilde{c}(p_2, s_{\mathit{bank}})$ *is given by* $(250,\ 270,\ 290)$ *dollars.*

Assume the fuzzy integration overhead cost is:

$$\tilde{\xi}(s_{\mathit{bank}}) = (50,\ 60,\ 70)\ \textit{dollars}.$$

Then, using fuzzy addition (performed componentwise), the overall fuzzy cost is:

$$\tilde{C}(s_{\mathit{bank}}) = \tilde{c}(p_1, s_{\mathit{bank}}) \oplus \tilde{c}(p_2, s_{\mathit{bank}}) \oplus \tilde{\xi}(s_{\mathit{bank}})$$

$$= (300 + 250 + 50,\ 320 + 270 + 60,\ 340 + 290 + 70) = (600,\ 650,\ 700)\ \textit{dollars}.$$

Thus, the online banking service has an overall fuzzy quality of approximately $(0.875,\ 0.90,\ 0.925)$ *and a fuzzy cost of* $(600,\ 650,\ 700)$ *dollars.*

Example 2.12 (Public Utility Payment Service via FSIM). *Consider a public utility payment service, denoted s_{payment}, which is delivered by two providers p_1 and p_2:*

$$M(s_{\text{payment}}) = \{p_1, p_2\}.$$

Assume the following fuzzy evaluations:

Fuzzy Quality:

- $\tilde{q}(p_1, s_{\text{payment}})$ *is represented by the triangular fuzzy number* $(0.80,\ 0.85,\ 0.90)$.
- $\tilde{q}(p_2, s_{\text{payment}})$ *is represented by* $(0.75,\ 0.80,\ 0.85)$.

Aggregating via the arithmetic mean, the overall fuzzy quality is:

$$\tilde{Q}(s_{\text{payment}}) \approx \left(\frac{0.80 + 0.75}{2},\ \frac{0.85 + 0.80}{2},\ \frac{0.90 + 0.85}{2}\right) = (0.775,\ 0.825,\ 0.875).$$

Fuzzy Cost:

- $\tilde{c}(p_1, s_{\text{payment}})$ *is given by* $(150,\ 160,\ 170)$ *dollars.*
- $\tilde{c}(p_2, s_{\text{payment}})$ *is given by* $(140,\ 150,\ 160)$ *dollars.*

Assume the fuzzy integration overhead cost is:

$$\tilde{\xi}(s_{\text{payment}}) = (20,\ 25,\ 30) \text{ dollars}.$$

Then the overall fuzzy cost is computed by:

$$\tilde{C}(s_{\text{payment}}) = (150 + 140 + 20,\ 160 + 150 + 25,\ 170 + 160 + 30) = (310,\ 335,\ 360) \text{ dollars}.$$

Thus, the public utility payment service is characterized by an overall fuzzy quality of $(0.775,\ 0.825,\ 0.875)$ *and an overall fuzzy cost of* $(310,\ 335,\ 360)$ *dollars.*

Theorem 2.13. *The Mathematical Framework for Fuzzy Service Integration and Management, $\mathcal{FSIM}$, generalizes the conventional Mathematical Framework for Service Integration and Management by incorporating fuzzy uncertainty. Specifically, if every fuzzy number $\tilde{q}(p, s)$, $\tilde{c}(p, s)$, and $\tilde{\xi}(s)$ degenerate to a singleton (i.e., $\tilde{q}(p, s) = \{q(p, s)\}$, etc.), then $\mathcal{FSIM}$ reduces exactly to the conventional framework. Moreover, by construction, the fuzzy functions used in $\mathcal{FSIM}$ exhibit the structure of fuzzy sets.*

Proof. Assume that for every service $s \in S$ and every provider $p \in M(s)$, the fuzzy quality and cost values are degenerate fuzzy numbers, that is,

$$\tilde{q}(p, s) = \{q(p, s)\}, \quad \tilde{c}(p, s) = \{c(p, s)\}, \quad \text{and} \quad \tilde{\xi}(s) = \{\xi(s)\}.$$

Then the fuzzy aggregation operator $\tilde{\varphi}$ becomes the standard aggregation operator φ, and fuzzy addition $\oplus$ reduces to standard addition. Consequently, for each service s:

$$\tilde{Q}(s) = \Big\{\varphi\big(\{q(p, s) : p \in M(s)\}\big)\Big\} = \{Q(s)\},$$

$$\tilde{C}(s) = \Big\{\sum_{p \in M(s)} c(p, s) + \xi(s)\Big\} = \{C(s)\}.$$

Thus, the FSIM tuple simplifies to:

$$\mathcal{FSIM} = \Big(S,\, P,\, M,\, \{q(p,s)\},\, \{c(p,s)\},\, \varphi,\, \xi\Big),$$

which is exactly the conventional Mathematical Framework for Service Integration and Management.

Furthermore, by definition, the mappings $\tilde{q} : S \times P \to \tilde{P}([0,1])$, $\tilde{c} : S \times P \to \tilde{P}(\mathbb{R}^+)$, and $\tilde{\xi} : S \to \tilde{P}(\mathbb{R}^+)$ are fuzzy set mappings. They assign to each element of their domain a fuzzy number (a set of possible values with degrees of membership), thereby endowing $\mathcal{FSIM}$ with a fuzzy set structure.

This proves that $\mathcal{FSIM}$ generalizes the conventional framework and possesses the inherent structure of fuzzy sets. □

Theorem 2.14 (Degeneracy Theorem). *If for every service $s \in S$ and each provider $p \in M(s)$ the fuzzy numbers degenerate to singletons, i.e.,*

$$\tilde{q}(p,s) = \{q(p,s)\}, \quad \tilde{c}(p,s) = \{c(p,s)\}, \quad \tilde{\xi}(s) = \{\xi(s)\},$$

then the FSIM framework reduces exactly to the conventional Mathematical Framework for Service Integration and Management, where

$$Q(s) = \varphi\big(\{q(p,s) : p \in M(s)\}\big) \quad \textit{and} \quad C(s) = \sum_{p \in M(s)} c(p,s) + \xi(s).$$

Proof. Assume for every $s \in S$ and $p \in M(s)$,

$$\tilde{q}(p,s) = \{q(p,s)\}, \quad \tilde{c}(p,s) = \{c(p,s)\}, \quad \tilde{\xi}(s) = \{\xi(s)\}.$$

Since the fuzzy aggregation operator $\tilde{\varphi}$ is defined to operate on sets of fuzzy numbers, when each set consists of a single element, $\tilde{\varphi}$ reduces to the standard aggregation operator φ. That is,

$$\tilde{Q}(s) = \tilde{\varphi}\Big(\{\tilde{q}(p,s) : p \in M(s)\}\Big) = \{\varphi(\{q(p,s) : p \in M(s)\})\} = \{Q(s)\}.$$

Similarly, fuzzy addition $\oplus$ on singletons reduces to ordinary addition, so that

$$\tilde{C}(s) = \bigoplus_{p \in M(s)} \tilde{c}(p,s) \oplus \tilde{\xi}(s) = \left\{ \sum_{p \in M(s)} c(p,s) + \xi(s) \right\} = \{C(s)\}.$$

Thus, the FSIM tuple becomes

$$\mathcal{FSIM} = \Big(S,\, P,\, M,\, \{q(p,s)\}_{p \in M(s),\, s \in S},\, \{c(p,s)\}_{p \in M(s),\, s \in S},\, \varphi,\, \xi\Big),$$

which is exactly the conventional framework for Service Integration and Management. □

Theorem 2.15 (Monotonicity of the Fuzzy Aggregation Operator). *Assume that for a given service $s \in S$ and for every provider $p \in M(s)$ the fuzzy quality values satisfy*

$$\tilde{q}_1(p,s) \leq \tilde{q}_2(p,s),$$

in the sense that the defuzzified (center-of-gravity) value of $\tilde{q}_1(p,s)$ is less than or equal to that of $\tilde{q}_2(p,s)$. If the fuzzy aggregation operator $\tilde{\varphi}$ is monotonic (i.e., it preserves this ordering), then

$$\tilde{Q}_1(s) = \tilde{\varphi}\Big(\{\tilde{q}_1(p,s) : p \in M(s)\}\Big) \leq \tilde{\varphi}\Big(\{\tilde{q}_2(p,s) : p \in M(s)\}\Big) = \tilde{Q}_2(s),$$

meaning the overall fuzzy quality for s obtained via $\tilde{q}_1$ is less than or equal to that obtained via $\tilde{q}_2$.

Proof. Let $q_1^*(p,s)$ and $q_2^*(p,s)$ denote the defuzzified (e.g., centroid) values of $\tilde{q}_1(p,s)$ and $\tilde{q}_2(p,s)$ respectively, and assume $q_1^*(p,s) \le q_2^*(p,s)$ for all $p \in M(s)$. Monotonicity of the fuzzy aggregation operator $\tilde{\varphi}$ implies that if every element in the set $\{\tilde{q}_1(p,s) : p \in M(s)\}$ is less than or equal to the corresponding element in $\{\tilde{q}_2(p,s) : p \in M(s)\}$ (in terms of defuzzified values), then

$$\tilde{\varphi}\Big(\{\tilde{q}_1(p,s) : p \in M(s)\}\Big) \le \tilde{\varphi}\Big(\{\tilde{q}_2(p,s) : p \in M(s)\}\Big).$$

This ordering is preserved when we defuzzify the aggregated fuzzy quality. Hence, the overall quality measurement $\tilde{Q}(s)$ is monotonic with respect to the individual fuzzy quality inputs. This completes the proof. □

2.3 *Neutrosophic Service Integration and Management*

Neutrosophic Service Integration and Management generalizes SIAM by representing quality, cost, integration with neutrosophic triples capturing truth, indeterminacy, and falsity. The definitions and related concepts of Neutrosophic Service Integration and Management are provided below.

Definition 2.16 (Mathematical Framework for Neutrosophic Service Integration and Management)**.** *To incorporate uncertainty more richly, we extend the above framework to the neutrosophic domain. In the* Neutrosophic Service Integration and Management *(NSIM) framework, uncertainty is modeled using neutrosophic numbers—that is, triples capturing degrees of truth, indeterminacy, and falsity.*
Specifically, we define:

- *The sets S, P, and the mapping $M : S \to \mathcal{P}(P) \setminus \{\emptyset\}$ remain as in the conventional framework.*
- *For each service $s \in S$ and each provider $p \in M(s)$, a* neutrosophic quality *is assigned:*

$$\tilde{q}_N(p,s) \in NS([0,1]),$$

where a neutrosophic number is a triple

$$\tilde{q}_N(p,s) = \big(T_q(p,s),\ I_q(p,s),\ F_q(p,s)\big),$$

with $T_q(p,s), I_q(p,s), F_q(p,s) \in [0,1]$ and

$$0 \le T_q(p,s) + I_q(p,s) + F_q(p,s) \le 3.$$

- *Similarly, for cost we define a* neutrosophic cost*:*

$$\tilde{c}_N(p,s) \in NS(\mathbb{R}^+),$$

and for the integration overhead,

$$\tilde{\xi}_N(s) \in NS(\mathbb{R}^+).$$

- *Let $\tilde{\varphi}_N$ be a neutrosophic aggregation operator that combines a set of neutrosophic quality numbers into a single neutrosophic quality measure.*

Thus, the NSIM framework is modeled as the tuple

$$\mathcal{NSIM} = \Big(S,\ P,\ M,\ \{\tilde{q}_N(p,s)\}_{p\in M(s),\, s\in S},\ \{\tilde{c}_N(p,s)\}_{p\in M(s),\, s\in S},\ \tilde{\varphi}_N,\ \tilde{\xi}_N\Big).$$

For each service $s \in S$, we define the overall neutrosophic quality and cost as:

$$\tilde{Q}_N(s) = \tilde{\varphi}_N\Big(\{\tilde{q}_N(p,s) : p \in M(s)\}\Big),$$

$$\tilde{C}_N(s) = \bigoplus_{p \in M(s)} \tilde{c}_N(p,s) \oplus \tilde{\xi}_N(s),$$

where $\bigoplus$ denotes the neutrosophic addition operation on neutrosophic numbers.

This framework generalizes conventional Service Integration and Management by representing quality, cost, and overhead as neutrosophic numbers—thereby explicitly modeling uncertainty (truth), indeterminacy, and falsity.

Example 2.17 (Cloud Hosting Service via Neutrosophic Service Integration and Management)**.** *Consider a cloud hosting service s_1 provided by two vendors, p_1 and p_2, so that:*

$$M(s_1) = \{p_1, p_2\}.$$

Neutrosophic Quality: *Assume the quality of service provided by each vendor is assessed as a neutrosophic number:*

- $\tilde{q}_N(p_1, s_1) = \big(0.85,\ 0.10,\ 0.05\big)$,
- $\tilde{q}_N(p_2, s_1) = \big(0.80,\ 0.15,\ 0.05\big)$.

Using a neutrosophic aggregation operator (for example, an average of the truth values while appropriately combining indeterminacy and falsity), we obtain the overall neutrosophic quality for s_1:

$$\tilde{Q}_N(s_1) \approx \big(0.825,\ 0.125,\ 0.05\big).$$

Neutrosophic Cost: *Assume the cost incurred by each vendor is given by:*

- $\tilde{c}_N(p_1, s_1) = \big(180,\ 0.05,\ 0.15\big)$ *dollars,*
- $\tilde{c}_N(p_2, s_1) = \big(190,\ 0.05,\ 0.15\big)$ *dollars.*

Let the fuzzy (neutrosophic) integration overhead be:

$$\tilde{\xi}_N(s_1) = \big(30,\ 0.10,\ 0.10\big) \textit{ dollars}.$$

Assuming neutrosophic addition is performed componentwise, the overall neutrosophic cost is:

$$\tilde{C}_N(s_1) = \big(180 + 190 + 30,\ 0.05 + 0.05 + 0.10,\ 0.15 + 0.15 + 0.10\big) = (400,\ 0.20,\ 0.40).$$

Here, the first component represents the aggregated cost; the second and third components reflect the aggregated degrees of indeterminacy and falsity in cost estimations.

Interpretation: *This example demonstrates a cloud hosting service where the overall quality is expressed as a neutrosophic number $(0.825, 0.125, 0.05)$ (indicating high quality with some uncertainty) and the total cost is represented as $(400, 0.20, 0.40)$ dollars. The use of neutrosophic numbers enables the decision-maker to capture not only the estimated cost and quality but also the uncertainty (indeterminacy) and error (falsity) associated with these estimates.*

If the neutrosophic evaluations were to collapse to single values with no uncertainty (i.e., indeterminacy equal to 0, and falsity equal to $1-$ truth), then this framework would reduce to the conventional Fuzzy Service Integration and Management model.

Example 2.18 (Online Banking Service via NSIM). *Consider an online banking service s_{bank} provided by two vendors, p_1 and p_2:*

$$M(s_{\text{bank}}) = \{p_1, p_2\}.$$

Neutrosophic Quality: *Assume the quality of each provider is evaluated as follows:*

$$\tilde{q}_N(p_1, s_{\text{bank}}) = (0.92, 0.05, 0.03), \quad \tilde{q}_N(p_2, s_{\text{bank}}) = (0.88, 0.07, 0.05).$$

Using a neutrosophic aggregation operator (for instance, an arithmetic mean on the truth components while appropriately combining the indeterminacy and falsity), the overall neutrosophic quality is approximately:

$$\tilde{Q}_N(s_{\text{bank}}) \approx \left(\frac{0.92 + 0.88}{2}, \text{aggregated } I, \text{aggregated } F\right) = (0.90, 0.06, 0.04).$$

Neutrosophic Cost: *Assume the cost incurred by each vendor is given by:*

$$\tilde{c}_N(p_1, s_{\text{bank}}) = (320, 0.03, 0.07) \text{ dollars}, \quad \tilde{c}_N(p_2, s_{\text{bank}}) = (280, 0.04, 0.06) \text{ dollars}.$$

Let the fuzzy integration overhead be:

$$\tilde{\xi}_N(s_{\text{bank}}) = (40, 0.02, 0.04) \text{ dollars}.$$

Then, using neutrosophic addition (componentwise addition on the crisp parts while combining the uncertainty components accordingly), the overall neutrosophic cost is:

$$\tilde{C}_N(s_{\text{bank}}) \approx (320 + 280 + 40, 0.03 + 0.04 + 0.02, 0.07 + 0.06 + 0.04) = (640, 0.09, 0.17) \text{ dollars}.$$

This evaluation reflects an overall quality of approximately $(0.90, 0.06, 0.04)$ and a total cost of around 640 dollars with the indicated uncertainty levels.

Example 2.19 (CRM System via NSIM). *Customer Relationship Management (CRM) is a comprehensive strategy for managing and analyzing customer interactions to enhance relationships and business outcomes (cf.[(32)–(34)]). Consider a Customer Relationship Management (CRM) system s_{CRM} provided by two vendors p_1 and p_2:*

$$M(s_{\text{CRM}}) = \{p_1, p_2\}.$$

Neutrosophic Quality: *Suppose:*

$$\tilde{q}_N(p_1, s_{\text{CRM}}) = (0.85, 0.12, 0.03),$$

$$\tilde{q}_N(p_2, s_{\text{CRM}}) = (0.80, 0.10, 0.10).$$

Using an appropriate neutrosophic aggregation, we obtain:

$$\tilde{Q}_N(s_{\text{CRM}}) \approx \left(\frac{0.85 + 0.80}{2}, \text{aggregated } I, \text{aggregated } F\right) = (0.825, 0.11, 0.065).$$

Neutrosophic Cost: *Assume:*

$$\tilde{c}_N(p_1, s_{\text{CRM}}) = (150, 0.04, 0.06) \text{ dollars},$$

$$\tilde{c}_N(p_2, s_{\text{CRM}}) = (140, 0.05, 0.05) \text{ dollars}.$$

Let the integration overhead be:

$$\tilde{\xi}_N(s_{\mathsf{CRM}}) = (20,\ 0.03,\ 0.02)\ \textit{dollars}.$$

Then the overall neutrosophic cost is computed as:

$$\tilde{C}_N(s_{\mathsf{CRM}}) = (150 + 140 + 20,\ 0.04 + 0.05 + 0.03,\ 0.06 + 0.05 + 0.02)$$

$$= (310,\ 0.12,\ 0.13)\ \textit{dollars}.$$

Thus, the CRM system achieves an overall neutrosophic quality of about $(0.825,\ 0.11,\ 0.065)$ *and an overall cost of approximately* $(310,\ 0.12,\ 0.13)$ *dollars.*

Example 2.20 (Online Gaming Service via NSIM)**.** *An online gaming system is a platform enabling interactive digital games, often in real time, globally connecting players and competitions [(35), (36)]. Consider an online gaming service* $s_{\textit{game}}$ *offered by three providers* p_1, p_2, *and* p_3*:*

$$M(s_{\textit{game}}) = \{p_1,\ p_2,\ p_3\}.$$

Neutrosophic Quality: *Assume the following quality evaluations (triangular neutrosophic numbers):*

$$\tilde{q}_N(p_1, s_{\textit{game}}) = (0.90,\ 0.05,\ 0.05),$$

$$\tilde{q}_N(p_2, s_{\textit{game}}) = (0.85,\ 0.10,\ 0.05),$$
$$\tilde{q}_N(p_3, s_{\textit{game}}) = (0.88,\ 0.07,\ 0.05).$$

Using a simple average for the truth components (and combining the uncertainty components appropriately), the overall quality is approximated as:

$$\tilde{Q}_N(s_{\textit{game}}) \approx \left(\frac{0.90 + 0.85 + 0.88}{3},\ \textit{aggregated } I,\ \textit{aggregated } F\right) \approx (0.8767,\ 0.07,\ 0.05).$$

Neutrosophic Cost: *Suppose the cost evaluations are:*

$$\tilde{c}_N(p_1, s_{\textit{game}}) = (200,\ 0.05,\ 0.05),$$

$$\tilde{c}_N(p_2, s_{\textit{game}}) = (180,\ 0.04,\ 0.06),$$
$$\tilde{c}_N(p_3, s_{\textit{game}}) = (190,\ 0.06,\ 0.04).$$

Assume the integration overhead is:

$$\tilde{\xi}_N(s_{\textit{game}}) = (50,\ 0.03,\ 0.02).$$

Then, the overall neutrosophic cost is:

$$\tilde{C}_N(s_{\textit{game}}) = (200 + 180 + 190 + 50,\ 0.05 + 0.04 + 0.06 + 0.03,$$

$$0.05 + 0.06 + 0.04 + 0.02) = (620,\ 0.18,\ 0.17).$$

This shows that the online gaming service has an overall neutrosophic quality of approximately $(0.877,\ 0.07,\ 0.05)$ *and a total cost of about* $(620,\ 0.18,\ 0.17)$ *dollars.*

Theorem 2.21. *The Mathematical Framework for Neutrosophic Service Integration and Management ($\mathcal{NSIM}$) generalizes the conventional (fuzzy) Service Integration and Management framework. Specifically, if for every $s \in S$ and each provider $p \in M(s)$ the neutrosophic values degenerate to singletons with no indeterminacy (i.e.,*

$$\tilde{q}_N(p, s) = \{(q(p, s),\, 0,\, 1 - q(p, s))\},$$
$$\tilde{c}_N(p, s) = \{(c(p, s),\, 0,\, 1 - c(p, s))\},$$
$$\tilde{\xi}_N(s) = \{(\xi(s),\, 0,\, 1 - \xi(s))\}\,)$$

then $\mathcal{NSIM}$ reduces exactly to the conventional Mathematical Framework for Service Integration and Management. Moreover, by their very construction, the mappings $\tilde{q}_N$, $\tilde{c}_N$, and $\tilde{\xi}_N$ exhibit the structure of neutrosophic sets.

Proof. Assume that for every $s \in S$ and $p \in M(s)$ the neutrosophic values are degenerate, so that

$$\tilde{q}_N(p, s) = \{(q(p, s),\, 0,\, 1 - q(p, s))\}, \quad \tilde{c}_N(p, s) = \{(c(p, s),\, 0,\, 1 - c(p, s))\},$$

and

$$\tilde{\xi}_N(s) = \{(\xi(s),\, 0,\, 1 - \xi(s))\}.$$

In this case, the neutrosophic aggregation operator $\tilde{\varphi}_N$ becomes equivalent to the standard (crisp) aggregation operator φ, and the neutrosophic addition $\bigoplus$ reduces to ordinary addition. Thus, for each service $s \in S$ we have:

$$\tilde{Q}_N(s) = \Big\{\varphi\big(\{q(p, s) : p \in M(s)\}\big)\Big\} = \{Q(s)\},$$
$$\tilde{C}_N(s) = \Big\{\sum_{p \in M(s)} c(p, s) + \xi(s)\Big\} = \{C(s)\}.$$

Hence, the NSIM tuple simplifies to:

$$\mathcal{NSIM} = \Big(S,\, P,\, M,\, \{q(p, s)\},\, \{c(p, s)\},\, \varphi,\, \xi\Big),$$

which is exactly the conventional framework.

Furthermore, by definition, the mappings $\tilde{q}_N : S \times P \to NS([0, 1])$, $\tilde{c}_N : S \times P \to NS(\mathbb{R}^+)$, and $\tilde{\xi}_N : S \to NS(\mathbb{R}^+)$ assign to each input a neutrosophic number (a triple (T, I, F) satisfying the neutrosophic conditions). Thus, the NSIM framework inherently possesses the structure of a neutrosophic set.

This completes the proof. □

Theorem 2.22 (Truth-Component Boundedness)**.** *Let $\tilde{\varphi}_N$ be a neutrosophic aggregation operator which, when applied to any nonempty finite collection of truth-values $\{T_1, \ldots, T_k\}$, satisfies*

$$\min_{1 \le i \le k} T_i \;\le\; T\big(\tilde{\varphi}_N(\{(T_i, I_i, F_i)\})\big) \;\le\; \max_{1 \le i \le k} T_i.$$

Then for each service $s \in S$, if

$$\tilde{q}_N(p, s) = (T_q(p, s), I_q(p, s), F_q(p, s)) \quad \big(p \in M(s)\big),$$

and

$$\tilde{Q}_N(s) = \tilde{\varphi}_N\big(\{\tilde{q}_N(p, s)\}\big) = (T_Q(s), I_Q(s), F_Q(s)),$$

we have

$$\min_{p \in M(s)} T_q(p, s) \;\le\; T_Q(s) \;\le\; \max_{p \in M(s)} T_q(p, s).$$

Proof. Set

$$\mathcal{T} = \{\, T_q(p, s) \mid p \in M(s)\}.$$

Since $M(s) \neq \emptyset$, $\mathcal{T}$ is nonempty. By definition of $\tilde{\varphi}_N$,

$$T_Q(s) = T\big(\tilde{\varphi}_N(\{\tilde{q}_N(p, s)\})\big) = \varphi_T(\mathcal{T}),$$

where φ_T is the truth-component aggregator. By the stated property of φ_T,

$$\min \mathcal{T} \;\le\; \varphi_T(\mathcal{T}) \;\le\; \max \mathcal{T}.$$

Hence

$$\min_{p \in M(s)} T_q(p, s) \;\le\; T_Q(s) \;\le\; \max_{p \in M(s)} T_q(p, s),$$

as required. □

Theorem 2.23 (Monotonicity of Truth and Indeterminacy)**.** *Suppose $\tilde{\varphi}_N$ is monotonic in its truth and indeterminacy components: if two collections of neutrosophic triples*

$$\{(T_i, I_i, F_i)\} \quad \text{and} \quad \{(T_i', I_i', F_i')\}$$

satisfy

$$T_i \le T_i' \quad \text{and} \quad I_i \le I_i' \quad \text{for all } i,$$

then writing

$$\tilde{\varphi}_N(\{(T_i, I_i, F_i)\}) = (T, I, F), \quad \tilde{\varphi}_N(\{(T_i', I_i', F_i')\}) = (T', I', F'),$$

we have

$$T \le T', \quad I \le I'.$$

Then in NSIM, if for a fixed service s each provider's neutrosophic evaluations improve in truth and indeterminacy:

$$T_q(p, s) \le T_q'(p, s), \quad I_q(p, s) \le I_q'(p, s) \quad (\forall\, p \in M(s)),$$

while falsities may correspondingly adjust, the aggregated values satisfy

$$T_Q(s) \le T_Q'(s), \quad I_Q(s) \le I_Q'(s),$$

where $\tilde{Q}_N(s) = (T_Q(s), I_Q(s), F_Q(s))$ and $\tilde{Q}'_N(s) = (T_Q'(s), I_Q'(s), F_Q'(s))$.

Proof. Let

$$X = \{\tilde{q}_N(p, s) \mid p \in M(s)\}, \quad X' = \{\tilde{q}_N'(p, s) \mid p \in M(s)\}.$$

By hypothesis,

$$T_q(p, s) \le T_q'(p, s), \quad I_q(p, s) \le I_q'(p, s) \quad (\forall\, p).$$

Applying the monotonicity of $\tilde{\varphi}_N$ yields

$$\tilde{\varphi}_N(X) = (T_Q, I_Q, F_Q) \;\le_{\mathrm{n}}\; \tilde{\varphi}_N(X') = (T_Q', I_Q', F_Q'),$$

which by definition means

$$T_Q \le T_Q', \quad I_Q \le I_Q'.$$

This establishes the desired monotonicity in the truth and indeterminacy components. □

Theorem 2.24 (Degenerate Reduction to Fuzzy SIAM)**.** *If for every $s \in S$ and $p \in M(s)$ the neutrosophic evaluations collapse to*

$$\tilde{q}_N(p,s) = \big(q(p,s),\ 0,\ 1 - q(p,s)\big), \quad \tilde{c}_N(p,s) = \big(c(p,s),\ 0,\ 1 - c(p,s)\big),$$

and likewise $\tilde{\xi}_N(s) = (\xi(s), 0, 1 - \xi(s))$, then NSIM reduces exactly to the original (fuzzy) SIAM framework.

Proof. Under the collapse $I \equiv 0$, $F \equiv 1 - T$, each neutrosophic triple $(T, 0, 1 - T)$ encodes precisely the fuzzy membership $T \in [0, 1]$. Moreover:

- Neutrosophic aggregation

  $$\tilde{\varphi}_N\big(\{(q_i, 0, 1 - q_i)\}\big) = \big(\varphi(\{q_i\}),\ 0,\ 1 - \varphi(\{q_i\})\big)$$

  projects to $\varphi(\{q_i\})$.

- Neutrosophic addition

  $$(c, 0, 1 - c) \oplus (c', 0, 1 - c') = \big(c + c',\ 0,\ 1 - (c + c')\big)$$

  projects to ordinary sum $c + c'$.

Applying these reductions to all quality, cost, and overhead terms shows that

$$\tilde{Q}_N(s) \ \mapsto\ Q(s), \quad \tilde{C}_N(s) \ \mapsto\ C(s),$$

and the NSIM tuple becomes identical with the SIAM tuple. Hence NSIM degenerates to SIAM. □

2.4 Mathematical Framework for ITIL 4 Framework

ITIL 4 is the latest evolution of the ITIL framework, emphasizing value co-creation, flexibility, and holistic service management practices (cf.[(37), (38)]). The definitions and related concepts of Mathematical Framework for ITIL 4 Framework are provided below.

Definition 2.25 (ITIL 4 Framework)**.** *We define the ITIL 4 framework as a 5-tuple*

$$\textit{ITIL4} = (\mathcal{G},\ \mathcal{C},\ T,\ \mathcal{P},\ I),$$

with the following components:

1. ***Guiding Principles:***

   $$\mathcal{G} = \{g_1, g_2, \ldots, g_r\}.$$

   Each guiding principle g_i is modeled as a predicate function

   $$g_i : \mathbb{R} \to \{0, 1\},$$

   so that for any quantitative measure x, $g_i(x) = 1$ indicates that the value x adheres to the principle (for example, "Focus on Value") and $g_i(x) = 0$ otherwise.

2. ***Governance Constraints***:

$$\mathcal{C} = \{c_1, c_2, \ldots, c_s\},$$

where each constraint c_j is expressed as an inequality on a service quality metric $q \in [0, 100]$. For example, one fundamental constraint is:

$$c_1: \quad q \geq 80.$$

To compute q, we assume a function

$$q = f(y),$$

where $y \in \mathbb{R}$ is the output produced by the service value chain.

3. ***Service Value Chain***: *The transformation function*

$$T : \mathbb{R}^n \to \mathbb{R}^m$$

represents the service value chain that converts an input resource vector $x \in \mathbb{R}^n$ into an output $y \in \mathbb{R}^m$. In ITIL 4, T is implemented by the sequential application (composition) of a set of practices:

$$T(x) = (p_k \circ p_{k-1} \circ \cdots \circ p_1)(x).$$

4. ***Practices***: *The set of practices is defined by*

$$\mathcal{P} = \{p_1, p_2, \ldots, p_k\},$$

where each practice p_i is a function:

$$p_i : \mathbb{R}^{n_i} \to \mathbb{R}^{m_i}.$$

Each p_i adds operational functionality and value; their composition yields the overall transformation T.

5. ***Continual Improvement***: *The continual improvement function is defined as*

$$I : \mathbb{N}_0 \to \mathbb{R},$$

mapping a discrete time index $t \in \mathbb{N}_0$ (measured in months, for instance) to an improvement increment $I(t)$, with the properties:

$$I(0) = 0, \quad \text{and } I(t) > 0 \text{ for } t > 0.$$

Thus, an improved service value after time t is given by:

$$V_{\textit{improved}}(t) = V + I(t),$$

where $V = T(x)$ is the base value resulting from the transformation.

Additionally, the framework is designed to maximize a service value function

$$v : \mathbb{R}^m \to \mathbb{R}_{\geq 0},$$

so that

$$v(y) = v\big(T(x)\big) \quad \text{subject to: } g_i(x) = 1 \;\forall\, g_i \in \mathcal{G} \text{ and } c_j\big(f(y)\big) \text{ holds } \forall\, c_j \in \mathcal{C}.$$

Example 2.26 (Concrete ITIL 4 Model)**.** *We now specify a concrete instance of the ITIL 4 framework.*

1. ***Guiding Principle:*** *Let*

$$\mathcal{G} = \{g_1\}, \quad \text{with} \quad g_1(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x \leq 0. \end{cases}$$

   This ensures that the input resource x must be positive ("Focus on Value").

2. ***Governance Constraint:*** *Define*

$$\mathcal{C} = \{c_1\} \quad \text{with} \quad c_1 : q \geq 80.$$

   Let the service quality be given by the function:

$$q = f(y) = \min\Big\{100,\ \frac{y}{4}\Big\}.$$

   Thus, to meet c_1 we require:

$$\frac{y}{4} \geq 80 \quad \Longrightarrow \quad y \geq 320.$$

3. ***Service Value Chain (Transformation Function):*** *Suppose the transformation is composed of two practices:*

$$T(x) = p_2\big(p_1(x)\big).$$

   Define:

$$p_1(x) = x + 20 \quad \text{and} \quad p_2(z) = 2z.$$

   Then the overall transformation is:

$$T(x) = 2(x + 20) = 2x + 40.$$

4. ***Continual Improvement:*** *Let the improvement function be linear in time:*

$$I(t) = 5t, \quad t \in \mathbb{N}_0.$$

5. ***Value Function:*** *Assume the service value function is linear:*

$$v(y) = y.$$

Step-by-Step Calculation:

(a) Initial Attempt with $x = 50$:

1. *Check the guiding principle: Since $50 > 0$, we have $g_1(50) = 1$ (condition satisfied).*

2. *Compute the transformed value:*

$$p_1(50) = 50 + 20 = 70.$$

$$p_2(70) = 2 \times 70 = 140.$$

   Thus,

$$T(50) = 140.$$

3. *Evaluate quality:*

$$q = \min\Big\{100,\ \frac{140}{4}\Big\} = \min\{100, 35\} = 35.$$

Since $35 < 80$*, the governance constraint* c_1 *is violated.*

(b) Adjustment to Satisfy the Constraint:

1. *In order to meet the quality constraint* $q \geq 80$*, we require:*

$$2x + 40 \geq 320 \quad \Longrightarrow \quad 2x \geq 280 \quad \Longrightarrow \quad x \geq 140.$$

2. *Choose an adjusted input* $x = 150$ *(which satisfies* $x \geq 140$*).*

3. *Recalculate the transformation:*

$$p_1(150) = 150 + 20 = 170.$$

$$p_2(170) = 2 \times 170 = 340.$$

Therefore,

$$T(150) = 340.$$

4. *Evaluate quality:*

$$q = \min\Big\{100,\ \frac{340}{4}\Big\} = \min\{100, 85\} = 85.$$

Now $85 \geq 80$*, so the governance constraint* c_1 *is met.*

(c) Applying Continual Improvement:

1. *For* $t = 3$ *months,*

$$I(3) = 5 \times 3 = 15.$$

2. *The final improved service value is:*

$$v_{\mathit{final}} = T(150) + I(3) = 340 + 15 = 355.$$

Thus, with an adjusted input $x = 150$ *and after 3 months of continual improvement, the ITIL 4 framework produces a service value of* 355 *while satisfying all guiding principles and governance constraints.*

Theorem 2.27 (Associativity of the Service Value Chain)**.** *If the service value chain is defined by the composition of practices*

$$T(x) \ = \ p_k\big(p_{k-1}(\cdots p_1(x)\cdots)\big),$$

then for any re-grouping of the composition the result is identical. In particular, for $1 \leq i < j \leq k$*,*

$$\big(p_k \circ \cdots \circ p_{j+1}\big) \circ \big(p_j \circ \cdots \circ p_i\big) \circ \big(p_{i-1} \circ \cdots \circ p_1\big) \ = \ p_k \circ \cdots \circ p_1.$$

Proof. Function composition is associative: for any three functions f, g, h of compatible types,

$$f \circ (g \circ h) = (f \circ g) \circ h.$$

One applies this identity repeatedly to re-associate the k-fold composition

$$p_k \circ \big(p_{k-1} \circ (\cdots \circ (p_1))\big)$$

into any desired grouping. Since each reassociation preserves the overall mapping, all parenthesizations yield the same $T(x)$. □

Theorem 2.28 (Monotonicity of the Transformation)**.** *Suppose each practice*

$$p_i : \mathbb{R}^{n_i} \to \mathbb{R}^{m_i}$$

is non-decreasing in each coordinate (i.e. $x \leq y$ componentwise implies $p_i(x) \leq p_i(y)$). Then the composite transformation

$$T = p_k \circ p_{k-1} \circ \cdots \circ p_1$$

is also non-decreasing: if $x \leq x'$ in $\mathbb{R}^n$ then $T(x) \leq T(x')$ in $\mathbb{R}^m$.

Proof. We prove by induction on the number of practices. For $k = 1$, $T = p_1$ is non-decreasing by assumption. Suppose the result holds for $k-1$, so

$$T_{k-1} = p_{k-1} \circ \cdots \circ p_1$$

is non-decreasing. Then

$$T(x) = p_k\big(T_{k-1}(x)\big),$$

and if $x \leq x'$ then $T_{k-1}(x) \leq T_{k-1}(x')$ by the induction hypothesis, and hence

$$T(x) = p_k\big(T_{k-1}(x)\big) \ \leq \ p_k\big(T_{k-1}(x')\big) = T(x')$$

by monotonicity of p_k. Thus T is non-decreasing for all k. □

Theorem 2.29 (Convexity of the Feasible Input Set)**.** *Suppose each guiding principle $g_i : \mathbb{R}^n \to \{0, 1\}$ and each governance constraint $c_j(f(T(x)))$ are given by linear inequalities:*

$$g_i(x) = 1 \iff a_i^T x \geq b_i, \quad c_j(f(T(x))) \iff d_j^T T(x) \geq e_j.$$

Then the set of all $x \in \mathbb{R}^n$ satisfying $g_i(x) = 1$ for all i and $c_j(f(T(x)))$ for all j is a convex polyhedron.

Proof. Each condition $a_i^T x \geq b_i$ defines a closed halfspace in $\mathbb{R}^n$, which is convex. Likewise, $d_j^T T(x) \geq e_j$ is equivalent to

$$(d_j^T \circ T)(x) \geq e_j,$$

and since T is affine (composition of affine p_i), $d_j^T T(x) \geq e_j$ defines another halfspace. The intersection of finitely many convex halfspaces is convex and polyhedral. Therefore the feasible input set

$$\{\, x : a_i^T x \geq b_i,\ d_j^T T(x) \geq e_j \ \forall\, i, j\}$$

is a convex polyhedron. □

Theorem 2.30 (Monotonic Continual Improvement)**.** *Let the base service value be $V = T(x) \in \mathbb{R}$ and the improvement function $I : \mathbb{N}_0 \to \mathbb{R}$ satisfy*

$$I(0) = 0, \quad I(t) > 0 \text{ for } t > 0.$$

Define the improved value

$$V_{\text{improved}}(t) = V + I(t).$$

Then $V_{\text{improved}}(t)$ is strictly increasing in $t \in \mathbb{N}_0$.

Proof. For $t = 0$, $V_{\text{improved}}(0) = V$. For any $t \geq 0$, since $I(t+1) > I(t) \geq 0$, we have

$$V_{\text{improved}}(t+1) = V + I(t+1) > V + I(t) = V_{\text{improved}}(t).$$

Thus each increment in t yields a strictly larger improved value. □

Theorem 2.31 (Affinity of the Service Value Chain). *If each practice $p_i \colon \mathbb{R}^{n_i} \to \mathbb{R}^{m_i}$ is an affine map*

$$p_i(x) = A_i x + b_i,$$

with matrix A_i and vector b_i, then the composite transformation

$$T(x) \;=\; (p_k \circ p_{k-1} \circ \cdots \circ p_1)(x)$$

is itself affine. In fact,

$$T(x) \;=\; A\,x + b,$$

where

$$A \;=\; A_k\,A_{k-1}\cdots A_1, \qquad b \;=\; A_k A_{k-1}\cdots A_2\,b_1 \;+\; A_k A_{k-1}\cdots A_3\,b_2 \;+\; \cdots \;+\; b_k.$$

Proof. We proceed by induction on the number of practices k.

Base case ($k = 1$). Trivial, since $T = p_1$ is affine by hypothesis.

Inductive step. Suppose that for $k-1$ practices

$$T_{k-1}(x) = A'\,x + b',$$

where

$$A' = A_{k-1}\cdots A_1, \quad b' = \sum_{i=1}^{k-1}\Big(A_{k-1}\cdots A_{i+1}\,b_i\Big).$$

Then

$$T(x) = p_k\big(T_{k-1}(x)\big) = A_k(A'\,x + b') + b_k = (A_k A')x + (A_k b' + b_k).$$

Setting $A = A_k A'$ and $b = A_k b' + b_k$ yields the claimed formula. This completes the induction. □

Theorem 2.32 (Convexity of the Objective under Affine Transformation). *Let $T \colon \mathbb{R}^n \to \mathbb{R}^m$ be the affine transformation from the previous theorem, and let $v \colon \mathbb{R}^m \to \mathbb{R}$ be a convex function. Then the composite function*

$$x \;\longmapsto\; v\big(T(x)\big)$$

is convex on $\mathbb{R}^n$.

Proof. By definition, $T(x) = Ax + b$ for some matrix A and vector b. For any $x, y \in \mathbb{R}^n$ and $\lambda \in [0,1]$,

$$T\big(\lambda x + (1-\lambda)y\big) \;=\; A\big(\lambda x + (1-\lambda)y\big) + b = \lambda\big(Ax + b\big) + (1-\lambda)\big(Ay + b\big) \;=\; \lambda T(x) + (1-\lambda)T(y).$$

Since v is convex,

$$v\big(\lambda T(x) + (1-\lambda)T(y)\big) \;\le\; \lambda\, v\big(T(x)\big) + (1-\lambda)\, v\big(T(y)\big).$$

Therefore,

$$v\big(T(\lambda x + (1-\lambda)y)\big) \;\le\; \lambda\, v\big(T(x)\big) + (1-\lambda)\, v\big(T(y)\big),$$

establishing convexity of $v \circ T$. □

Theorem 2.33 (Existence of an Optimal Solution). *Assume*

- *the feasible set*

  $$\mathcal{F} = \big\{x \in \mathbb{R}^n \;\big|\; g_i(x) = 1\;\forall\, i,\; c_j\big(f(T(x))\big) \text{ holds } \forall\, j\big\}$$

  is nonempty, closed, and bounded (hence compact),

- *the objective function* $v \circ T$ *is continuous.*

Then there exists at least one $x^* \in \mathcal{F}$ *at which*

$$v\big(T(x^*)\big) = \max_{x \in \mathcal{F}} v\big(T(x)\big).$$

Proof. By Weierstrass's theorem, any continuous real-valued function on a nonempty compact set attains its maximum. Here, $\mathcal{F}$ is compact by hypothesis and $v \circ T$ is continuous as a composition of continuous functions (T is affine hence continuous, and v is continuous). Therefore there exists $x^* \in \mathcal{F}$ such that

$$v\big(T(x^*)\big) = \max_{x \in \mathcal{F}} v\big(T(x)\big).$$

This completes the proof. □

2.5 Fuzzy ITIL 4 Framework

The Fuzzy ITIL 4 Framework represents guiding principles, constraints, practices, transformation, and improvement with fuzzy degrees, enabling nuanced, graded confidence. The definitions and related concepts of Mathematical Framework for Fuzzy ITIL 4 Framework are provided below (cf.[(39)–(41)]).

Definition 2.34 (Fuzzy ITIL 4 Framework)**.** *The* Fuzzy ITIL 4 Framework *generalizes the crisp ITIL 4 Framework by associating fuzzy membership values to each component. It is defined as the quintuple*

$$\widetilde{ITIL4} = \Big(\widetilde{\mathcal{G}}, \widetilde{\mathcal{C}}, \widetilde{T}, \widetilde{\mathcal{P}}, \widetilde{I}\Big),$$

with the following fuzzified components:

1. ***Fuzzy Guiding Principles:*** *Let* $\mathcal{U}_G$ *be the universe of all candidate guiding principles. Then* $\widetilde{\mathcal{G}}$ *is a fuzzy subset of* $\mathcal{U}_G$ *with membership function*

$$\mu_{\widetilde{\mathcal{G}}} : \mathcal{U}_G \to [0, 1].$$

A value of $\mu_{\widetilde{\mathcal{G}}}(g) = 1$ *indicates full adherence of principle* g*; values in* $(0, 1)$ *reflect partial support.*

2. ***Fuzzy Governance Constraints:*** *Assume the crisp governance constraints in* $\mathcal{C}$ *are conditions on a quality metric* q*. In the fuzzy model, the degree to which* q *satisfies the constraint is given by the function*

$$\mu_{\widetilde{\mathcal{C}}} : \mathbb{R} \to [0, 1].$$

For example, one may define

$$\mu_{\widetilde{\mathcal{C}}}(q) = \begin{cases} 0, & q < 70, \\ \dfrac{q - 70}{10}, & 70 \le q \le 80, \\ 1, & q > 80. \end{cases}$$

3. ***Fuzzy Service Value Chain Transformation:*** *The crisp transformation* $T : X \to Y$ *is extended to a fuzzy transformation* $\widetilde{T}$*. For each input* $x \in X$*, rather than outputting a single value* $T(x)$*, we associate a fuzzy set in* Y*. In many practical cases, one expresses*

$$\widetilde{T}(x) = T(x) \quad \textit{with an associated confidence degree} \quad \mu_{\widetilde{T}}(x) \in [0, 1].$$

4. ***Fuzzy Practices:*** *The set of practices is fuzzified as*

$$\widetilde{\mathcal{P}} = \{\widetilde{p}_1, \widetilde{p}_2, \ldots, \widetilde{p}_k\},$$

where each fuzzy practice $\widetilde{p}_i : Z_i \to W_i$ *is equipped with a membership function*

$$\mu_{\widetilde{p}_i} : W_i \to [0, 1].$$

Their sequential (fuzzy) composition yields the overall transformation $\widetilde{T}$.

5. ***Fuzzy Continual Improvement:*** *The improvement function is extended to the fuzzy domain as* $\widetilde{I} : \mathbb{N}_0 \to \mathbb{R}$. *Often the improvement is assumed to be fully confident, i.e., its associated degree is 1.*

Furthermore, a fuzzy service value function

$$\widetilde{v} : Y \to [0, 1],$$

assigns to each output a fuzzy satisfaction level.

Example 2.35 (Concrete Instance of the Fuzzy ITIL 4 Framework). *Consider the following instance:*
1. ***Fuzzy Guiding Principle:*** *Let the universe* $\mathcal{U}_G$ *include the following principles:*

$$\mathcal{U}_G = \{\text{“Focus on Value”},\ \text{“Collaborate and Promote Visibility”},\ \text{others}\}.$$

Define the membership function:

$$\mu_{\widetilde{G}}(g) = \begin{cases} 1, & \text{if } g = \text{“Focus on Value”}, \\ 0.8, & \text{if } g = \text{“Collaborate and Promote Visibility”}, \\ 0.5, & \text{otherwise}. \end{cases}$$

2. ***Fuzzy Governance Constraint:*** *Let a quality metric* q *(with* $0 \leq q \leq 100$*) be measured by the function*

$$\mu_{\widetilde{C}}(q) = \begin{cases} 0, & q < 70, \\ \dfrac{q-70}{10}, & 70 \leq q \leq 80, \\ 1, & q > 80. \end{cases}$$

For instance, if $q = 85$, *then* $\mu_{\widetilde{C}}(85) = 1$ *(fully satisfied).*
3. ***Fuzzy Service Value Chain Transformation:*** *Assume the crisp transformation is given by*

$$T(x) = 2(x + 20).$$

Its fuzzy extension is represented by a confidence degree:

$$\mu_{\widetilde{T}}(x) = \begin{cases} 0.7, & x < 100, \\ 1, & x \geq 100. \end{cases}$$

For an input $x = 150$, *we have*

$$T(150) = 2(150 + 20) = 340 \quad \text{and} \quad \mu_{\widetilde{T}}(150) = 1.$$

4. Fuzzy Practice: *Let there be a single fuzzy practice* $\widetilde{p}_1$ *defined by*

$$\widetilde{p}_1(x) = x + 20,$$

with an associated membership function (using a logistic model)

$$\mu_{\widetilde{p}_1}(x) = \frac{1}{1 + e^{-0.1(x-50)}}.$$

For $x = 150$*, we calculate* $\widetilde{p}_1(150) = 170$ *and find that* $\mu_{\widetilde{p}_1}(150) \approx 0.98$*, indicating high confidence.*

5. Fuzzy Continual Improvement: *Define the fuzzy improvement function as*

$$\widetilde{I}(t) = 5t, \quad t \in \mathbb{N}_0,$$

with full confidence (i.e., degree 1).

Step-by-Step Computation:

1. *For input* $x = 150$*, the crisp transformation yields*

$$T(150) = 2(150 + 20) = 340.$$

With $\mu_{\widetilde{T}}(150) = 1$*, the fuzzy transformation acts fully.*

2. *Suppose the quality metric is computed by*

$$q = \min\{100,\ 340/4\} = \min\{100,\ 85\} = 85.$$

Therefore,

$$\mu_{\widetilde{C}}(85) = 1,$$

meaning the governance constraint is completely satisfied.

3. *The fuzzy practice outputs:*

$$\widetilde{p}_1(150) = 150 + 20 = 170,$$

with $\mu_{\widetilde{p}_1}(150) \approx 0.98.$

4. *After* $t = 3$ *months, the fuzzy improvement is*

$$\widetilde{I}(3) = 5 \times 3 = 15.$$

5. *The final output is then:*

$$y_{\textit{final}} = T(150) + \widetilde{I}(3) = 340 + 15 = 355.$$

6. *The fuzzy service value function is computed as*

$$\widetilde{v}(355) = \frac{355}{355 + 50} \approx 0.876.$$

This example demonstrates how fuzzification adds graded confidence and satisfaction levels to every stage of the ITIL 4 Framework, while still retaining the structure of the original crisp model.

Theorem 2.36 (Reduction to the Crisp ITIL 4 Framework)**.** *If all the membership functions in the Fuzzy ITIL 4 Framework*

$$\widetilde{ITIL4} = \left(\widetilde{\mathcal{G}},\, \widetilde{\mathcal{C}},\, \widetilde{T},\, \widetilde{\mathcal{P}},\, \widetilde{I}\right)$$

are replaced by their characteristic functions (i.e., restricted to $\{0,1\}$*), then*

$$\widetilde{ITIL4} \equiv (\mathcal{G},\, \mathcal{C},\, T,\, \mathcal{P},\, I).$$

Proof. Replacing a membership function $\mu : U \to [0,1]$ by its characteristic function χ_U means that for every $u \in U$

$$\chi_U(u) = \begin{cases} 1, & \text{if } u \text{ satisfies the crisp condition}, \\ 0, & \text{otherwise}. \end{cases}$$

Then:

- The fuzzy guiding principles $\widetilde{\mathcal{G}}$ become the crisp set $\mathcal{G}$ since for every $g \in \mathcal{U}_G$ we have $\chi_{\widetilde{\mathcal{G}}}(g) \in \{0,1\}$.
- The fuzzy governance constraint $\mu_{\widetilde{\mathcal{C}}}(q)$ yields 1 when q meets the constraint and 0 otherwise, thereby reproducing the crisp constraint $\mathcal{C}$.
- The fuzzy transformation $\widetilde{T}(x)$ is equivalent to $T(x)$ when the associated confidence degree $\mu_{\widetilde{T}}(x)$ is either 0 or 1.
- Each fuzzy practice $\widetilde{p}_i$ reduces to its crisp counterpart p_i if $\mu_{\widetilde{p}_i}(w) \in \{0,1\}$ for all w.
- The fuzzy continual improvement $\widetilde{I}$ becomes I when its improvement degree is binary.

Therefore, if every fuzzy component is degenerated to its binary (crisp) counterpart, the Fuzzy ITIL 4 Framework reduces exactly to the traditional ITIL 4 Framework. □

Theorem 2.37 (Min–Composition Property)**.** *Let*

$$\widetilde{T}(x) = \widetilde{p}_k\big(\widetilde{p}_{k-1}(\cdots \widetilde{p}_1(x)\cdots)\big)$$

be the fuzzy transformation obtained by composing k *fuzzy practices. If each practice* $\widetilde{p}_i$ *has membership function* μ_i*, and we define*

$$\mu_{\widetilde{T}}(x) = \min_{1 \le i \le k} \mu_i(x_i),$$

where $x_1 = x$ *and* $x_{i+1} = \widetilde{p}_i(x_i)$*, then*

$$\mu_{\widetilde{T}}(x) = \min\big\{\mu_1(x_1), \mu_2(x_2), \ldots, \mu_k(x_k)\big\}.$$

Proof. We prove by induction on k.

Base case: For $k = 1$, $\widetilde{T} = \widetilde{p}_1$ and the definition gives

$$\mu_{\widetilde{T}}(x) = \mu_1(x) = \min\{\mu_1(x)\},$$

so the claim holds.

Inductive step: Assume the result holds for $(k-1)$ practices. Write

$$\widetilde{T}_{k-1}(x) = \widetilde{p}_{k-1}\big(\cdots \widetilde{p}_1(x)\cdots\big), \quad \mu_{\widetilde{T}_{k-1}}(x) = \min_{1 \le i \le k-1} \mu_i(x_i).$$

Then for k practices,

$$\widetilde{T}(x) = \widetilde{p}_k\big(\widetilde{T}_{k-1}(x)\big),$$

and by the definition of $\min$–composition,

$$\mu_{\widetilde{T}}(x) = \min\big\{\mu_{\widetilde{T}_{k-1}}(x),\, \mu_k(x_k)\big\} = \min\Big\{\min_{1\le i\le k-1} \mu_i(x_i),\, \mu_k(x_k)\Big\} = \min_{1\le i\le k} \mu_i(x_i).$$

This completes the induction. □

Theorem 2.38 (Monotonicity of the Fuzzy Transformation). *Suppose each fuzzy practice $\widetilde{p}_i$ has a membership function μ_i that is non-decreasing in its argument: for all w, w',*

$$w \le w' \quad \Longrightarrow \quad \mu_i(w) \le \mu_i(w').$$

Then for any inputs $x \le x'$, the composite transformation $\widetilde{T}$ satisfies

$$\mu_{\widetilde{T}}(x) \;\le\; \mu_{\widetilde{T}}(x').$$

Proof. Write $x_1 = x,\; x_1' = x'$, and recursively $x_{i+1} = \widetilde{p}_i(x_i)$, $x_{i+1}' = \widetilde{p}_i(x_i')$. Since each $\widetilde{p}_i$ is non-decreasing,

$$x_i \le x_i' \quad \Longrightarrow \quad \mu_i(x_i) \le \mu_i(x_i').$$

By the Min–Composition Property,

$$\mu_{\widetilde{T}}(x) = \min_{1\le i\le k} \mu_i(x_i) \;\le\; \min_{1\le i\le k} \mu_i(x_i') = \mu_{\widetilde{T}}(x'),$$

establishing monotonicity. □

Theorem 2.39 (Bounds Preservation). *Let $\alpha > 0$ be a lower bound such that for every component of the Fuzzy ITIL 4 Framework,*

$$\mu_{\widetilde{\mathcal{G}}}(g),\; \mu_{\widetilde{\mathcal{C}}}(q),\; \mu_i(w),\; \mu_{\widetilde{T}}(x),\; \mu_{\widetilde{I}}(t) \;\ge\; \alpha.$$

Then the final fuzzy service value $\widetilde{v}(y)$ also satisfies

$$\mu_{\widetilde{v}}(y) \;\ge\; \alpha.$$

Proof. Both $\min$ and any convex combination (e.g. arithmetic mean) of values in $[\alpha, 1]$ remain in $[\alpha, 1]$. Since $\widetilde{v}(y)$ is obtained by applying such aggregation operators to inputs each at least α, it follows that

$$\mu_{\widetilde{v}}(y) \ge \alpha.$$

Concretely, if

$$\widetilde{v}(y) = \lambda \cdot \min_i r_i + (1-\lambda) \cdot \frac{1}{m} \sum_j s_j,$$

with each $r_i, s_j \in [\alpha, 1]$ and $\lambda \in [0, 1]$, then

$$\mu_{\widetilde{v}}(y) \ge \lambda\alpha + (1-\lambda)\alpha = \alpha.$$

Hence the lower bound is preserved. □

2.6 Neutrosophic ITIL 4 Framework

The Neutrosophic ITIL 4 Framework encodes all components—principles, constraints, practices, transformation, and improvement—as neutrosophic triples of truth, indeterminacy, and falsity. The definitions and related concepts of Mathematical Framework for Neutrosophic ITIL 4 Framework are provided below.

Definition 2.40 (Neutrosophic ITIL 4 Framework)**.** *The* Neutrosophic ITIL 4 Framework *generalizes the Fuzzy ITIL 4 Framework by endowing every component with a neutrosophic structure. It is defined as the quintuple*

$$\widetilde{NITIL4} = \Big(\widetilde{\mathcal{G}}_N,\ \widetilde{\mathcal{C}}_N,\ \widetilde{T}_N,\ \widetilde{\mathcal{P}}_N,\ \widetilde{I}_N\Big),$$

with the following components:

1. ***Neutrosophic Guiding Principles:*** *Let $\mathcal{U}_G$ be the universe of candidate guiding principles. Then the neutrosophic set of guiding principles is defined by a triple membership function*

$$\eta_{\widetilde{\mathcal{G}}} : \mathcal{U}_G \to [0,1]^3,$$

where for each $g \in \mathcal{U}_G$,

$$\eta_{\widetilde{\mathcal{G}}}(g) = \big(T_{\widetilde{\mathcal{G}}}(g),\ I_{\widetilde{\mathcal{G}}}(g),\ F_{\widetilde{\mathcal{G}}}(g)\big).$$

2. ***Neutrosophic Governance Constraints:*** *Suppose a quality metric $q \in \mathbb{R}$ is used to assess a constraint. Its neutrosophic satisfaction is given by*

$$\eta_{\widetilde{\mathcal{C}}}(q) = \big(T_{\widetilde{\mathcal{C}}}(q),\ I_{\widetilde{\mathcal{C}}}(q),\ F_{\widetilde{\mathcal{C}}}(q)\big),$$

where—for example—we may specify

$$T_{\widetilde{\mathcal{C}}}(q) = \begin{cases} 0, & q < 70, \\ \frac{q-70}{10}, & 70 \le q \le 80, \\ 1, & q > 80, \end{cases}$$

and assign suitable indeterminacy $I_{\widetilde{\mathcal{C}}}(q)$ (say, a small constant when q is near the threshold) and falsity $F_{\widetilde{\mathcal{C}}}(q)$ such that

$$T_{\widetilde{\mathcal{C}}}(q) + I_{\widetilde{\mathcal{C}}}(q) + F_{\widetilde{\mathcal{C}}}(q) = 1 \quad \textit{or is otherwise appropriately bounded}.$$

3. ***Neutrosophic Service Value Chain Transformation:*** *Extend the crisp transformation $T : X \to Y$ to a neutrosophic mapping. For each $x \in X$, instead of a single output $T(x)$, we associate the neutrosophic value*

$$\eta_{\widetilde{T}}(x) = \Big(T_{\widetilde{T}}(x),\ I_{\widetilde{T}}(x),\ F_{\widetilde{T}}(x)\Big) \in [0,1]^3.$$

In many practical cases one may define

$$\widetilde{T}(x) = T(x) \quad \textit{with confidence } \eta_{\widetilde{T}}(x).$$

4. ***Neutrosophic Practices:*** *Let*

$$\widetilde{\mathcal{P}}_N = \{\widetilde{p}_1,\ \widetilde{p}_2, \ldots,\ \widetilde{p}_k\},$$

where each neutrosophic practice $\widetilde{p}_i : Z_i \to W_i$ is paired with a triple membership function

$$\eta_{\widetilde{p}_i} : W_i \to [0,1]^3.$$

Their sequential composition, following neutrosophic aggregation rules, produces the overall transformation $\widetilde{T}_N$.

5. ***Neutrosophic Continual Improvement:*** *Extend the improvement function* $I : \mathbb{N}_0 \to \mathbb{R}$ *to*

$$\widetilde{I}_N : \mathbb{N}_0 \to \mathbb{R},$$

along with an associated neutrosophic confidence triple (often taken as $(1, 0, 0)$ *for full certainty).*

Finally, the neutrosophic service value function is given by

$$\eta_{\widetilde{v}} : Y \to [0,1]^3,$$

assigning to each output y *a triple* $(T_{\widetilde{v}}(y),\ I_{\widetilde{v}}(y),\ F_{\widetilde{v}}(y))$ *that represents the degree of satisfaction from the perspectives of truth, indeterminacy, and falsity.*

Example 2.41 (Concrete Instance of the Neutrosophic ITIL 4 Framework)**.** *Consider the following instance:*

1. ***Neutrosophic Guiding Principles:*** *Let the universe* $\mathcal{U}_G$ *include:*

$$\mathcal{U}_G = \{\text{“Focus on Value”},\ \text{“Collaborate and Promote Visibility”},\ \ldots\}.$$

Define the neutrosophic membership as:

$$\eta_{\widetilde{G}}(g) = \begin{cases} (1,\ 0,\ 0), & \text{if } g = \text{“Focus on Value”}, \\ (0.8,\ 0.1,\ 0.1), & \text{if } g = \text{“Collaborate and Promote Visibility”}, \\ (0.5,\ 0.2,\ 0.3), & \text{otherwise}. \end{cases}$$

2. ***Neutrosophic Governance Constraint:*** *For a quality metric* q *(with* $0 \le q \le 100$*), define:*

$$\eta_{\widetilde{C}}(q) = \begin{cases} (0,\ 0.1,\ 0.9), & q < 70, \\ \left(\frac{q-70}{10},\ 0.05,\ 1 - \frac{q-70}{10} - 0.05\right), & 70 \le q \le 80, \\ (1,\ 0,\ 0), & q > 80. \end{cases}$$

For example, if $q = 85$ *then*

$$\eta_{\widetilde{C}}(85) = (1,\ 0,\ 0),$$

indicating full satisfaction.

3. ***Neutrosophic Service Value Chain Transformation:*** *Assume the crisp transformation is*

$$T(x) = 2(x + 20).$$

Its neutrosophic extension is defined by

$$\eta_{\widetilde{T}}(x) = \begin{cases} (0.7,\ 0.2,\ 0.1), & x < 100, \\ (1,\ 0,\ 0), & x \ge 100. \end{cases}$$

For an input $x = 150$*,*

$$T(150) = 2(150 + 20) = 340 \quad \text{and} \quad \eta_{\widetilde{T}}(150) = (1, 0, 0).$$

4. ***Neutrosophic Practice:*** *Let there be one neutrosophic practice* $\widetilde{p}_1$ *defined as*

$$\widetilde{p}_1(x) = x + 20.$$

With an associated membership defined by a logistic-like model:

$$\eta_{\widetilde{p}_1}(x) = \Big(\frac{1}{1+e^{-0.1(x-50)}},\ 0.05,\ 1-\frac{1}{1+e^{-0.1(x-50)}}-0.05\Big).$$

For $x = 150$*, we obtain*

$$\widetilde{p}_1(150) = 170, \quad \eta_{\widetilde{p}_1}(150) \approx (0.98,\ 0.05,\ -0.03).$$

(Here, one may adjust the model so that the falsity is nonnegative; for instance, by calibrating the offset.)

5. Neutrosophic Continual Improvement: *Define the improvement function as*

$$\widetilde{I}_N(t) = 5t, \quad t \in \mathbb{N}_0,$$

with confidence given by $(1, 0, 0)$.

Step-by-Step Computation:

1. *For an input* $x = 150$*, the crisp transformation gives*

$$T(150) = 340, \quad \eta_{\widetilde{T}}(150) = (1, 0, 0).$$

2. *Compute a quality metric* q *by, for instance,*

$$q = \min\{100,\ 340/4\} = \min\{100,\ 85\} = 85.$$

Then,

$$\eta_{\widetilde{C}}(85) = (1, 0, 0),$$

meaning the governance constraint is fully met.

3. *The neutrosophic practice yields*

$$\widetilde{p}_1(150) = 150 + 20 = 170, \quad \eta_{\widetilde{p}_1}(150) \approx (0.98,\ 0.05,\ 0.02),$$

after adjusting the values to ensure $T + I + F \leq 1$ *(or a proper bound).*

4. *After* $t = 3$ *months,*

$$\widetilde{I}_N(3) = 5 \times 3 = 15 \quad \textit{with } (1, 0, 0).$$

5. *The final service output is calculated as*

$$y_{\textit{final}} = T(150) + \widetilde{I}_N(3) = 340 + 15 = 355.$$

6. *The neutrosophic service value function is defined as*

$$\eta_{\widetilde{v}}(y) = \Big(\frac{y}{y+50},\ 0.05,\ 1-\frac{y}{y+50}-0.05\Big).$$

Therefore, for $y = 355$,

$$\eta_{\widetilde{v}}(355) \approx \Big(\frac{355}{405},\ 0.05,\ 1-\frac{355}{405}-0.05\Big) \approx (0.877,\ 0.05,\ 0.073).$$

This example demonstrates how the Neutrosophic ITIL 4 Framework embeds uncertainty (via indeterminacy) and captures both truth and falsity degrees for each component. Moreover, by setting $I = 0$ and $F = 1 - T$, the model reverts to the Fuzzy ITIL 4 Framework, as established in the theorem.

Example 2.42 (Network Outage Incident Management via Neutrosophic ITIL 4 Framework)**.** *Consider an unplanned network outage incident in an enterprise IT environment. We apply the Neutrosophic ITIL 4 Framework to assess how well the incident management process adheres to principles, meets constraints, executes practices, and improves over time.*

1. Neutrosophic Guiding Principles *Let*

$$\mathcal{U}_G = \{\text{“Focus on Value”},$$

$$\text{“Collaborate and Promote Visibility”},\ \text{“Keep It Simple”}\}$$

. Assign:

$$\eta_{\widetilde{G}}(g) = \begin{cases} (1.00,\ 0.00,\ 0.00), & g = \text{“Focus on Value”}, \\ (0.90,\ 0.10,\ 0.00), & g = \text{“Collaborate and Promote Visibility”}, \\ (0.80,\ 0.15,\ 0.05), & g = \text{“Keep It Simple”}. \end{cases}$$

2. Neutrosophic Governance Constraint *Let q = resolution time in hours. Define*

$$T_{\widetilde{C}}(q) = \begin{cases} 1.0, & q \le 4, \\ \dfrac{6-q}{2}, & 4 < q \le 6, \\ 0.0, & q > 6, \end{cases} \quad I_{\widetilde{C}}(q) = \begin{cases} 0.0, & q \le 4, \\ 0.20, & 4 < q \le 6, \\ 0.10, & q > 6, \end{cases}$$

and

$$F_{\widetilde{C}}(q) = 1 - T_{\widetilde{C}}(q)\ -\ I_{\widetilde{C}}(q).$$

For an incident resolved in $q = 5$ h:

$$\eta_{\widetilde{C}}(5) = \big(0.50,\ 0.20,\ 0.30\big).$$

3. Neutrosophic Service Value Chain Transformation *The incident passes through three practices:* detection, resolution, *and* communication. *Their neutrosophic memberships are:*

$$\eta_{\widetilde{p}_1}(\textit{detection}) = (0.85,\ 0.10,\ 0.05),$$

$$\eta_{\widetilde{p}_2}(\textit{resolution}) = (0.80,\ 0.15,\ 0.05),$$

$$\eta_{\widetilde{p}_3}(\textit{communication}) = (0.90,\ 0.05,\ 0.05).$$

By the Neutrosophic Composition Rule,

$$\eta_{\widetilde{T}_N}(x) = \Big(\min\{0.85, 0.80, 0.90\},$$

$$\max\{0.10, 0.15, 0.05\},\ \max\{0.05, 0.05, 0.05\}\Big) = (0.80,\ 0.15,\ 0.05).$$

4. Combine with Governance Constraint *Intersecting with $\eta_{\widetilde{C}}(5) = (0.50, 0.20, 0.30)$:*

$$T = \min(0.80, 0.50) = 0.50, \quad I = \max(0.15, 0.20) = 0.20, \quad F = \max(0.05, 0.30) = 0.30.$$

Thus after value-chain and constraint:

$$\eta_1 = (0.50,\ 0.20,\ 0.30).$$

5. Neutrosophic Continual Improvement *Monthly post-incident review has* $\eta_{\widetilde{I}_N}(t) = (0.70,\ 0.20,\ 0.10)$. *Intersecting:*

$$T_{\textit{final}} = \min(0.50, 0.70) = 0.50, \quad I_{\textit{final}} = \max(0.20, 0.20) = 0.20, \quad F_{\textit{final}} = \max(0.30, 0.10) = 0.30.$$

So the final neutrosophic service value is

$$\eta_{\widetilde{v}} = (0.50,\ 0.20,\ 0.30).$$

This triple concisely captures that the incident process achieved 50% truth in desired outcomes, with 20% indeterminacy and 30% residual falsity, guiding management on where to focus further improvements.

Theorem 2.43 (Reduction to the Fuzzy ITIL 4 Framework)**.** *If, for every neutrosophic component of the Neutrosophic ITIL 4 Framework, the indeterminacy is set to zero and the falsity is defined as the complement of truth (i.e.* $F(u) = 1-T(u)$ *for all relevant* u*), then the Neutrosophic ITIL 4 Framework*

$$\widetilde{NITIL4} = \left(\widetilde{\mathcal{G}}_N,\ \widetilde{\mathcal{C}}_N,\ \widetilde{T}_N,\ \widetilde{\mathcal{P}}_N,\ \widetilde{I}_N\right)$$

reduces exactly to the Fuzzy ITIL 4 Framework.

Proof. For any neutrosophic membership function $\eta : U \to [0,1]^3$, suppose that for each $u \in U$ we set

$$I(u) = 0 \quad \text{and} \quad F(u) = 1 - T(u).$$

Then the triple $\eta(u) = (T(u),\ 0,\ 1 - T(u))$ encodes exactly the same information as the fuzzy membership function $\mu(u) = T(u)$. Applying this conversion to each component:

- The neutrosophic guiding principles $\eta_{\widetilde{\mathcal{G}}}(g) = (T_{\widetilde{\mathcal{G}}}(g), 0, 1 - T_{\widetilde{\mathcal{G}}}(g))$ reduce to the fuzzy set with membership $\mu_{\widetilde{\mathcal{G}}}(g) = T_{\widetilde{\mathcal{G}}}(g)$.
- Similarly, for the governance constraints, the neutrosophic satisfaction $\eta_{\widetilde{\mathcal{C}}}(q) = (T_{\widetilde{\mathcal{C}}}(q), 0, 1 - T_{\widetilde{\mathcal{C}}}(q))$ becomes the fuzzy membership $\mu_{\widetilde{\mathcal{C}}}(q) = T_{\widetilde{\mathcal{C}}}(q)$.
- The neutrosophic service value chain and practices, when their associated membership triples are replaced by $(T, 0, 1-T)$, correspond exactly to the fuzzy transformation and fuzzy practices.
- The neutrosophic continual improvement $\widetilde{I}_N$ similarly reduces to the fuzzy continual improvement I if its uncertainty is removed.

Hence, under these conditions, every component of $\widetilde{NITIL4}$ is equivalent to its fuzzy counterpart, and we obtain the Fuzzy ITIL 4 Framework. This completes the proof. □

Theorem 2.44 (Neutrosophic Composition Rule)**.** *Let each neutrosophic practice* $\widetilde{p}_i$ *be given by*

$$\eta_{\widetilde{p}_i}(w) = \big(T_i(w),\ I_i(w),\ F_i(w)\big),$$

and define the composite mapping

$$\widetilde{T}_N(x) = \widetilde{p}_k\big(\widetilde{p}_{k-1}(\dots \widetilde{p}_1(x) \dots)\big).$$

Then its neutrosophic membership is

$$\eta_{\widetilde{T}_N}(x) = \Big(\min_{1 \le i \le k} T_i(x_i),\ \max_{1 \le i \le k} I_i(x_i),\ \max_{1 \le i \le k} F_i(x_i)\Big),$$

where $x_1 = x$ *and* $x_{i+1} = \widetilde{p}_i(x_i)$.

Proof. We use induction on the number of practices k.

Base case ($k = 1$). Then $\widetilde{T}_N = \widetilde{p}_1$, so

$$\eta_{\widetilde{T}_N}(x) = \eta_{\widetilde{p}_1}(x) = \big(T_1(x), I_1(x), F_1(x)\big) = (\min\{T_1(x)\}, \max\{I_1(x)\}, \max\{F_1(x)\}).$$

Thus the formula holds for $k = 1$.

Inductive step. Assume the formula holds for $k - 1$:

$$\eta_{\widetilde{T}_N^{(k-1)}}(x) = \big(\min_{1\le i\le k-1} T_i(x_i),\ \max_{1\le i\le k-1} I_i(x_i),\ \max_{1\le i\le k-1} F_i(x_i)\big).$$

Now consider k practices. Write

$$y \ = \ \widetilde{T}_N^{(k-1)}(x), \quad \eta_{\widetilde{T}_N^{(k-1)}}(x) = \big(T', I', F'\big).$$

Then

$$\widetilde{T}_N(x) = \widetilde{p}_k(y), \quad \eta_{\widetilde{p}_k}(y) = \big(T_k(y), I_k(y), F_k(y)\big).$$

By the definition of neutrosophic intersection,

$$\eta_{\widetilde{T}_N}(x) = \eta_{\widetilde{T}_N^{(k-1)}} \wedge \eta_{\widetilde{p}_k} = (\min(T', T_k(y)),\ \max(I', I_k(y)),\ \max(F', F_k(y))).$$

Substitute $T' = \min_{i<k} T_i(x_i)$, $I' = \max_{i<k} I_i(x_i)$, $F' = \max_{i<k} F_i(x_i)$ to obtain

$$\big(\min_{i\le k} T_i(x_i),\ \max_{i\le k} I_i(x_i),$$

$$\max_{i\le k} F_i(x_i)\big),$$

completing the induction. □

Theorem 2.45 (Monotonicity of Neutrosophic Aggregation). *If for each practice i and all $w \le w'$,*

$$T_i(w) \le T_i(w'),$$

$$I_i(w) \le I_i(w'),$$

$$F_i(w) \ge F_i(w'),$$

then for any inputs $x \le x'$ componentwise,

$$T_{\widetilde{T}_N}(x) \le T_{\widetilde{T}_N}(x'),$$

$$I_{\widetilde{T}_N}(x) \le I_{\widetilde{T}_N}(x'),$$

$$F_{\widetilde{T}_N}(x) \ge F_{\widetilde{T}_N}(x').$$

Proof. Let $x_1 = x$, $x'_1 = x'$, and recursively

$$x_{i+1} = \widetilde{p}_i(x_i)$$

,

$$x'_{i+1} = \widetilde{p}_i(x'_i)$$

. Since each $\widetilde{p}_i$ is order-preserving,

$$x_i \le x'_i \quad \Longrightarrow$$

$$T_i(x_i) \le T_i(x_i'),\ I_i(x_i) \le I_i(x_i'),\ F_i(x_i) \ge F_i(x_i').$$

Then taking the minimum over T_i,

$$\min_i T_i(x_i) \le \min_i T_i(x_i'),$$

and the maximum over I_i and F_i,

$$\max_i I_i(x_i) \le \max_i I_i(x_i'), \quad \max_i F_i(x_i) \ge \max_i F_i(x_i').$$

But by the composition theorem,

$$T_{\widetilde{T}_N}(x) = \min_i T_i(x_i),$$
$$I_{\widetilde{T}_N}(x) = \max_i I_i(x_i),$$
$$F_{\widetilde{T}_N}(x) = \max_i F_i(x_i),$$

and similarly for x', yielding the desired inequalities. □

Theorem 2.46 (Exact Reduction to Fuzzy ITIL 4)**.** *If for every neutrosophic component*

$$\eta(u) = (T(u), I(u), F(u))$$

one sets

$$I(u) = 0, \quad F(u) = 1 - T(u),$$

then the entire Neutrosophic ITIL 4 Framework collapses to the Fuzzy ITIL 4 Framework.

Proof. Under $I(u) = 0$ and $F(u) = 1-T(u)$, each triple $(T, 0, 1-T)$ is in one-to-one correspondence with the single fuzzy membership $\mu(u) = T$. Moreover, neutrosophic intersection

$$(T_1, 0, 1 - T_1) \wedge (T_2, 0, 1 - T_2) = \big(\min(T_1, T_2),\ 0,\ 1 - \min(T_1, T_2)\big)$$

agrees exactly with the fuzzy $\min$–composition. Similarly, all aggregation steps (max for indeterminacy and falsity) reduce to trivial operations on zero and complement, reproducing the fuzzy definitions. Hence every neutrosophic set, relation, and operation becomes its fuzzy counterpart, establishing the equivalence of frameworks. □

3. Conclusion and Future Works

In this paper, we introduced two new frameworks—Fuzzy Service Integration and Management (FSIM) and Neutrosophic Service Integration and Management (NSIM)—which embed these uncertainty models into Service Integration and Management (SIAM). We also explored how ITIL best practices can be fused with fuzzy and neutrosophic logic to create more adaptive and resilient service-management processes.

As future work, we plan to investigate extended models based on Plithogenic Sets[(42)–(44)], HeptaPartitioned Neutrosophic Sets[(45), (46)], HyperNeutrosophic Sets[(47)–(49)], and QuadriPartitioned Neutrosophic Sets[(50), (51)]. Additionally, we aim to explore graph-based[(52)], bidirected graph-based[(53)–(56)], hypergraph-based[(57)–(59)], and superhypergraph-based[(60)–(63)] approaches to Service Integration and Management, depending on application requirements.

Conflict of Interest

The author declares that there are no conflicts of interest related to this manuscript.

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