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Some Spherical Uncertain Linguistic Aggregation Operators and their Application in Multi-Attribute Decision-Making

Haolun Wang[1,*], Jiang Liu[1]

[1] School of Economics & Management, Nanchang Hankong University, Nanchang 330063, Jiangxi, China

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ABSTRACT

Motivated by the idea of uncertain linguistic sets and spherical fuzzy sets, the concept of spherical uncertain linguistic sets (SULSs) and some basic operational rules, properties, the score, and accuracy for spherical uncertain linguistic numbers (SULNs) are proposed. Then, we utilize arithmetic and geometric operations to develop some spherical uncertain linguistic aggregation operators: spherical uncertain linguistic weighted average (SULWA) operator, spherical uncertain linguistic weighted geometric (SULWG) operator, spherical uncertain linguistic ordered weighted average (SULOWA) operator, spherical uncertain linguistic ordered weighted geometric (SULOWG) operator, spherical uncertain linguistic hybrid average (SULHA) operator, spherical uncertain linguistic hybrid geometric (SULHG) operator, generalized spherical uncertain linguistic weighted averaging (GSULWA) operator, and generalized spherical uncertain linguistic weighted geometric (GSULWG) operator. Moreover, some desired properties of these operators are investigated. Then, we use these operators to develop some approaches to solve spherical uncertain linguistic multi-attribute decision-making problems. Finally, a practical example for smartphone selection is given to illustrate the developed approach, and comparison analysis and discussion are conducted with another existing method to demonstrate the effectiveness and feasibility of the developed approach.

1. Introduction

MADM problems are the important research parts of modern decision theory. Since the object things are fuzzy, the attribute values involved in the decision problems are not always expressed as crisp numbers, and some of them are more suitable to be denoted by fuzzy numbers, linguistic variable, and rough numbers etc. in many situations, a decision maker can't provide the information with a crisp value, but with a linguistic term because of time pressure, lack of knowledge or data, and his/her limited expertise related with problem domain. As a result, many decision making processes take place in the setting with linguistic information [1-8]. However, in some practical situations, the

* *Corresponding author.*
E-mail address: hlwang71162@nchu.edu.cn

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input linguistic arguments may not match any of the original linguistic labels, and they may be located between two of them. For instance, when evaluating the “design” of a car, an expert may provide his/her opinion with “between ‘fair’ and ‘good’ [9-10]. Xu [9] emphasized those linguistic methods failed in cope with the situations in which the information about preference values was expressed in the form of uncertain linguistic information. Therefore, the decision making problem with uncertain linguistic terms deserves more attention. There are two main types linguistic sets are called the additive uncertain linguistic set and the multiplicative uncertain linguistic set, respectively. In aspect of the additive uncertain linguistic set, many aggregation operators are proposed by scholars to aggregate the decision information based on the operational laws, such as the uncertain linguistic weighted averaging (ULWA) operator[11], the uncertain ordered weighted averaging (ULOWA) operator[11], the uncertain linguistic hybrid aggregation (ULHA) operator[12], some induced uncertain linguistic OWA (IULOWA) operators[12], the uncertain linguistic weighted harmonic mean (ULWHM) operator [13], the uncertain linguistic ordered weighted harmonic mean (ULOWHM) operator [13], the uncertain linguistic hybrid harmonic mean (ULHHM) operator [14], the uncertain linguistic Choquet averaging (ULCA) operator [14], the generalized Shapley uncertain linguistic Choquet averaging (GS-ULCA) operator [15] , the uncertain linguistic Bonferroni mean (ULBM) operator [15], the uncertain linguistic prioritized weighted averaging (ULPWA) operator [16] and the uncertain linguistic prioritized weighted harmonic averaging (ULPWHA) operator [17], etc. To aggregate the decision information in the form of multiplicative linguistic variables, many aggregation operators are proposed based on the multiplicative linguistic sets and their operational rules, such as the uncertain linguistic weighted geometric averaging (ULWG) operator [17], the uncertain linguistic ordered weighted geometric averaging (ULOWG) operator [18], the induced uncertain linguistic ordered weighted geometric averaging (IULOWG) operator [18], the uncertain linguistic hybrid geometric averaging (ULHG) operator [18], the uncertain multiplicative linguistic hybrid weighted geometric averaging (ULHWG) operator [19], the uncertain linguistic geometric Bonferroni mean (ULGBM) operator [15], the uncertain linguistic prioritized weighted geometric (ULPWG) operator [16]， the uncertain probabilistic linguistic geometric Bonferroni mean (UPLGBM) operator [20], and so on. In recent years, some evaluation scenarios extend application of uncertain linguistic set and make it adapt to a certain decision-making environment. For example, some researchers effectively integrated the uncertain linguistic set with intuitionistic fuzzy sets (IFSs) [21-23], interval-valued intuitionistic fuzzy sets (IVIFSs) [24-28], Pythagorean fuzzy sets (PyFSs) [29-31], picture fuzzy sets (PFSs) [32], 2-dimension linguistic [33-37], 2-tuple linguistic [38-39], q-rung orthopair fuzzy sets (q-ROFSs) [40-41], hesitant fuzzy sets (HFSs) [42-43], neutrosophic sets (NSs) [44-48], rough sets (RSs) [49], soft sets (SSs) [50], fuzzy soft sets (FSSs) [51], etc. These works extend the wider applications of uncertain linguistic set.

Spherical fuzzy sets (SFSs) proposed by Ashraf S. *et al.,* [52] and are extension of IFSs, PyFSs and PFSs. SFSs are more powerful than IFSs, PyFSs and PFSs in dealing with the uncertainty, imprecision, and vagueness. Compared with IFSs and PyFSs, SFSs are able to handle fuzzy concepts of Yes, Abstain, No and Refusal” in real life, such as voting, but IFSs and PyFSs can only handle fuzzy concepts such as “neither this nor that” and their ignore the “Abstain and Refusal ” in real situation, which is not sufficiently close to human nature. And compared with PFSs, the positive-membership degree $P(x)$, neutral-membership degree $I(x)$ and negative-membership degree $N(x)$ in SFSs satisfy the condition $(\forall x \in X)\left(0 \le (P(x))^2 + (I(x))^2 + (N(x))^2 \le 1\right)$. If the problem cannot be handled by using PFSs when $P(x)+I(x)+N(x)>1$, for example, the degree of memberships of an alternative are 0.2, 0.6 and 0.6, respectively, but their square sum is less than 1 with SFSs. Overall, the decision space of SFSs is greater than the decision space of PFSs, PyFSs and IFSs. Thus, SFSs could model some decision-making situations which IFSs, PyFSs and PFSs cannot deal with. From this point of view, inspired by the results [26,31,32], this paper generalized the concept of the spherical uncertain linguistic set. The aim of this study is to develop a series of spherical uncertain linguistic aggregation operators, including the spherical uncertain linguistic weighted average (SULWA) operator, spherical uncertain linguistic weighted geometric (SULWG) operator, spherical uncertain linguistic ordered weighted average (SULOWA) operator, spherical uncertain linguistic ordered weighted geometric (SULOWG) operator, spherical uncertain linguistic hybrid average (SULHA) operator, spherical uncertain linguistic hybrid geometric (SULHG) operator, generalized spherical uncertain linguistic weighted averaging (GSULWA) operator and generalized spherical uncertain linguistic weighted geometric (GSULWG) operator.

To do this, the remainder of this paper is structured as follows. In Section 2, some basic definitions about uncertain linguistic variable sets and spherical fuzzy sets are reviewed, and the concept of spherical uncertain linguistic sets (SULSs) and spherical uncertain linguistic numbers (SULNs) are defined, then score, accuracy, and operational rules are proposed. In Section 3 and Section 4, some operators are developed, such as SULWA operator, SULWG operator, SULOWA operator, SULOWG operator, SULHA operator, SULHG operator, GSULWA operator and GSULWG operator, and their basic properties are discussed. In Section 5, a MADM method is developed based SULHA, SULHG, GSULWA and GSULWG operators. In Section 6, an illustrative example about smart phone selection is provided, and comparison analysis and discussion are conducted to verify the effectiveness and feasibility of the developed approach. Some concludes and future research are given in the last section.

2 Preliminaries

2.1 Uncertain linguistic variable sets

Definition 1. Let $S=\{s_i \mid i=1,2,\ldots,,t\}$ be a linguistic term set with odd cardinality. Any label, s_i represents a possible value for a linguistic variable, and it should satisfy the following characteristics [53-56]:

The set is ordered: $s_i > s_j$, if $i > j$;

Max operator: $\max(s_i, s_j) = s_i$, if $s_i \geq s_j$;

Min operator: $\min(s_i, s_j) = s_i$, if $s_i \leq s_j$.

For example, S can be defined as S = {s1=extremely poor, s2=very poor, s3=poor, s4= medium, s5=good, s6=very good, s7=extremely good}.

Let $\tilde{s}=[s_\theta, s_\tau]$, where $s_\theta, s_\tau \in \tilde{S}$, s_θ and s_τ are the lower and upper limits, respectively. We call $\tilde{s}$ the uncertain linguistic variable. Let $\tilde{S}$ be the set of all the uncertain linguistic variable sets [11,17,57,58].

Let any three uncertain linguistic variables $\tilde{s}_i=\left[s_{\theta_i}, s_{\tau_i}\right]$, $\tilde{s}_j=\left[s_{\theta_j}, s_{\tau_j}\right]$ and $\tilde{s}_k=\left[s_{\theta_k}, s_{\tau_k}\right]$, $\tilde{s}_i, \tilde{s}_j, \tilde{s}_k \in \tilde{S}$, $\lambda \in [0,1]$, Xu [11] define their operational laws as follows:

$$\tilde{s}_i \oplus \tilde{s}_j=\left[s_{\theta_i}, s_{\tau_i}\right] \oplus\left[s_{\theta_j}, s_{\tau_j}\right]=\left[s_{\theta_i} \oplus s_{\theta_j}, s_{\tau_i} \oplus s_{\tau_j}\right]=\left[s_{\theta_i+\theta_j}, s_{\tau_i+\tau_j}\right] \tag{1}$$

$$\tilde{s}_i \otimes \tilde{s}_j=\left[s_{\theta_i}, s_{\tau_i}\right] \otimes\left[s_{\theta_j}, s_{\tau_j}\right]=\left[s_{\theta_i} \otimes s_{\theta_j}, s_{\tau_i} \otimes s_{\tau_j}\right]=\left[s_{\theta_i \times \theta_j}, s_{\tau_i \times \tau_j}\right] \tag{2}$$

$$\lambda \tilde{s}_k=\lambda\left[s_{\theta_k}, s_{\tau_k}\right]=\left[\lambda s_{\theta_k}, \lambda s_{\tau_k}\right]=\left[s_{\lambda \theta_k}, s_{\lambda \tau_k}\right] \tag{3}$$

$$\left(\tilde{s}_k\right)^\lambda=\left(\left[s_{\theta_k}, s_{\tau_k}\right]\right)^\lambda=\left[\left(s_{\theta_k}\right)^\lambda,\left(s_{\tau_k}\right) \lambda\right]=\left[s_{\left(\theta_k\right)^\lambda}, s_{\left(\tau_k\right)^\lambda}\right] \tag{4}$$

2.2 Spherical fuzzy sets

Spherical fuzzy sets (SFSs) [52] are extension of IFSs, PyFSs [59] and PFSs [60]. SFSs based models may be adequate in situations when we face human opinions involving more answers of types: Yes, Abstain, No and Refusal. It can be considered as a powerful tool represent the uncertain information in the process of patterns recognition and cluster analysis.

Definition 2. [61] Let the universe set be R. Then the set

$$A=\left\{\left\langle r,P_A(r),I_A(r),N_A(r)\middle| r\in R\right\rangle\right\} \tag{5}$$

is said to be SFS, where PA:R→[0,1], IA:R→[0,1] and NA:R→[0,1] are said to be the degree of positive-membership degree of r in R, neutral-membership degree of r in R and negative-membership degree of r in R respectively. Also PA, IA, and NA satisfy the following condition:

$$(\forall r\in R)\left(0\le\left(P_A(r)\right)^2+\left(I_A(r)\right)^2+\left(N_A(r)\right)^2\le 1\right) \tag{6}$$

For SFS $\left\{\left\langle r,P_A(r),I_A(r),N_A(r)\middle| r\in R\right\rangle\right\}$, which is triple components $\left\langle P_A(r),I_A(r),N_A(r)\right\rangle$ are said to SFN and each SFN can be denoted by $e=\left\langle P_e,I_e,N_e\right\rangle$, where Pe, Ie and Ne∈[0,1], with condition that $0\le P_e^2+I_e^2+N_e^2\le 1$.

Definition 3. [61] Assuming that $e_j=\left\langle P_{e_j},I_{e_j},N_{e_j}\right\rangle$ and $e_k=\left\langle P_{e_k},I_{e_k},N_{e_k}\right\rangle$ are any two SFNs. The union, intersection, and compliment are described as

$$e_j\subseteq e_k \text{ iff } \forall r\in R, P_{e_j}\le P_{e_k}, I_{e_j}\le I_{e_k}, N_{e_j}\ge N_{e_k}\ ; \tag{7}$$

$$e_j=e_k \text{ iff } e_j\subseteq e_k, e_k\subseteq e_j\ ; \tag{8}$$

$$e_j\cup e_k=\left\langle \max\left(P_{e_j},P_{e_k}\right),\min\left(I_{e_j},I_{e_k}\right),\min\left(N_{e_j},N_{e_k}\right)\right\rangle; \tag{9}$$

$$e_j\cap e_k=\left\langle \min\left(P_{e_j},P_{e_k}\right),\min\left(I_{e_j},I_{e_k}\right),\max\left(N_{e_j},N_{e_k}\right)\right\rangle; \tag{10}$$

$$e_j^c=\left\langle N_{e_j},I_{e_j},P_{e_j}\right\rangle \tag{11}$$

Definition 4. [61] Suppose that $e_j=\left\langle P_{e_j},I_{e_j},N_{e_j}\right\rangle$ and $e_k=\left\langle P_{e_k},I_{e_k},N_{e_k}\right\rangle$ are any two SFNs and $\tau\ge 0$. Then the operations of SFNs can be denotes as

$$\tau e_j=\left\langle \sqrt{1-\left(1-P_{e_j}^2\right)^{\tau}},\left(I_{e_j}\right)^{\tau},\left(N_{e_j}\right)^{\tau}\right\rangle; \tag{12}$$

$$e_j+e_k=\left\langle \sqrt{P_{e_j}^2+P_{e_k}^2-P_{e_j}^2\cdot P_{e_k}^2},I_{e_j}\cdot I_{e_k},N_{e_j}\cdot N_{e_k}\right\rangle; \tag{13}$$

$$e_j\times e_k=\left\langle P_{e_j}\cdot P_{e_k},I_{e_j}\cdot I_{e_k},\sqrt{N_{e_j}^2+N_{e_k}^2-N_{e_j}^2\cdot N_{e_k}^2}\right\rangle; \tag{14}$$

$$e_j^{\tau}=\left\langle \left(P_{e_j}\right)^{\tau},\left(I_{e_j}\right)^{\tau},\sqrt{1-\left(1-N_{e_j}^2\right)^{\tau}}\right\rangle. \tag{15}$$

Based on Definition 4, we can derive the following properties easily.

Theorem 1. Assuming that $e_i=\left\langle P_{e_i},I_{e_i},N_{e_i}\right\rangle$, $e_j=\left\langle P_{e_j},I_{e_j},N_{e_j}\right\rangle$ and $e_k=\left\langle P_{e_k},I_{e_k},N_{e_k}\right\rangle$ be any three SFNs and $\lambda\ge 0$. Then the following identities are satisfies.

$$e_i\oplus e_j=e_j\oplus e_i; \tag{16}$$

$$e_i\otimes e_j=e_j\otimes e_i; \tag{17}$$

$$\left(e_i\oplus e_j\right)\oplus e_k=e_i\oplus\left(e_j\oplus e_k\right); \tag{18}$$

$$\left(e_i\otimes e_j\right)\otimes e_k=e_i\otimes\left(e_j\otimes e_k\right); \tag{19}$$

$$\lambda e_i \oplus \lambda e_j = \lambda\left(e_i \oplus e_j\right); \tag{20}$$

$$\lambda_i e_i \oplus \lambda_j e_i = \left(\lambda_i \oplus \lambda_j\right) e_i; \tag{21}$$

$$\left(e_i \otimes e_j\right)^{\lambda} = e_i^{\lambda} \otimes e_j^{\lambda}; \tag{22}$$

$$e_j^{\lambda_i} \otimes e_j^{\lambda_j} = e_j^{\lambda_i+\lambda_j}; \tag{23}$$

2.3 Spherical uncertain linguistic sets

In the following, we shall propose the concepts and basic operations of the Spherical uncertain linguistic sets on the basis of the spherical fuzzy sets and uncertain linguistic information processing model.

Definition 5. A spherical uncertain linguistic sets A in X is given

$$A = \left\{\tilde{s}_{\theta(x)}, \left(P_{A(x)}, I_{A(x)}, N_{A(x)}\right), x \in X\right\} \tag{24}$$

Where $\tilde{s}_{\alpha(x)}\left(\tilde{s}_{\alpha(x)} = \left[s_{\theta(x)}, s_{\tau(x)}\right]\right)$, $s_{\theta(x)}$ and $s_{\tau(x)}$ are the lower and upper limits, respectively. $P_{A(x)} \in [0,1]$, $I_{A(x)} \in [0,1]$ and $N_{A(x)} \in [0,1]$, with the condition $0 \le \left(P_{A(x)}\right)^2 + \left(I_{A(x)}\right)^2 + \left(N_{A(x)}\right)^2 \le 1$. The numbers $P_{A(x)}$, $I_{A(x)}$, $N_{A(x)}$ represent, respectively, the degree of positive membership, degree of neutral membership and degree of negative membership of the element x to uncertain linguistic variable $\tilde{s}_{\alpha(x)}$. Then for $x \in X$, $\pi_{A(x)} = \sqrt{1-\left(\left(P_{A(x)}\right)^2 + \left(I_{A(x)}\right)^2 + \left(N_{A(x)}\right)^2\right)}$ could be called the degree of refusal membership of the element x to uncertain linguistic variable $\tilde{s}_{\alpha(x)}$.

For convenience, we call $\tilde{a} = \left\langle \tilde{s}_{\beta(a)}, \left(P_a, I_a, N_a\right)\right\rangle$ a spherical uncertain linguistic number (SULN), where $P_a \in [0,1]$, $I_a \in [0,1]$, $N_a \in [0,1]$, $0 \le \left(P_a\right)^2 + \left(I_a\right)^2 + \left(N_a\right)^2 \le 1$, $\tilde{s}_{\beta(a)}\left(\tilde{s}_{\beta(a)} = \left[s_{\theta(a)}, s_{\tau(a)}\right]\right)$.

Definition 6. Let $\tilde{a}_i = \left\langle \tilde{s}_{\alpha_i}, \left(P_i, I_i, N_i\right)\right\rangle = \left\langle \left[s_{\theta_i}, s_{\tau_i}\right], \left(P_i, I_i, N_i\right)\right\rangle$ be any SULNs. Then

A expectation value of a spherical uncertain linguistic number can be represented as follows:

$$EX\left(\tilde{a}_i\right) = \frac{\left(\theta_i + \tau_i\right) \times \left(2 + P_i - I_i - N_i\right)}{6t} \tag{25}$$

The accuracy value of a spherical uncertain linguistic number can be represented as follows:

$$AC\left(\tilde{a}_i\right) = \frac{\left(\theta_i + \tau_i\right) \times \left(P_i - N_i\right)}{2t} \tag{26}$$

Idea takes from Definition 6, is the technique which using for equating the SULNs can be described as

Theorem 2. Let $\tilde{a}_i = \left\langle \left[s_{\theta_i}, s_{\tau_i}\right], \left(P_i, I_i, N_i\right)\right\rangle$ and $\tilde{a}_j = \left\langle \left[s_{\theta_j}, s_{\tau_j}\right], \left(P_j, I_j, N_j\right)\right\rangle$ be any two SULNs. Then by using Definition 6, equating technique can be described as

If $EX\left(\tilde{a}_i\right) > EX\left(\tilde{a}_j\right)$, then $\tilde{a}_i > \tilde{a}_j$;

If $EX\left(\tilde{a}_i\right) = EX\left(\tilde{a}_j\right)$, and $AC\left(\tilde{a}_i\right) > AC\left(\tilde{a}_j\right)$, then $\tilde{a}_i > \tilde{a}_j$.

Motivated by the operations of the uncertain linguistic information and Definition 6, in the following, we shall define some operational laws of spherical uncertain linguistic numbers.

Definition 7. Let $\tilde{a}_i = \left\langle \left[s_{\theta_i}, s_{\tau_i}\right], \left(P_i, I_i, N_i\right)\right\rangle$ and $\tilde{a}_j = \left\langle \left[s_{\theta_j}, s_{\tau_j}\right], \left(P_j, I_j, N_j\right)\right\rangle$ be any two SULNs and $\lambda \ge 0$. Then

$$\tilde{a}_i \oplus \tilde{a}_j = \left\langle \left[s_{\theta_i} + s_{\theta_j}, s_{\tau_i} + s_{\tau_j} \right], \left(\sqrt{P_i^2 + P_j^2 - P_i^2 \cdot P_j^2}, I_i \cdot I_j, N_i \cdot N_j \right) \right\rangle \tag{27}$$

$$\tilde{a}_i \otimes \tilde{a}_j = \left\langle \left[s_{\theta_i} \cdot s_{\theta_j}, s_{\tau_i} \cdot s_{\tau_j} \right], \left(P_i \cdot P_j, I_i \cdot I_j, \sqrt{N_i^2 + N_j^2 + N_i^2 \cdot N_j^2} \right) \right\rangle \tag{28}$$

$$\lambda \tilde{a}_i = \left\langle \left[\lambda s_{\theta_i}, \lambda s_{\tau_i} \right], \left(\sqrt{1 - \left(1 - P_i^2\right)^{\lambda}}, \left(I_i\right)^{\lambda}, \left(N_i\right)^{\lambda} \right) \right\rangle \tag{29}$$

$$\left(\tilde{a}_i\right)^{\lambda} = \left\langle \left[\left(s_{\theta_i}\right)^{\lambda}, \left(s_{\tau_i}\right)^{\lambda} \right], \left(\left(P_i\right)^{\lambda}, \left(I_i\right)^{\lambda}, \sqrt{1 - \left(1 - N_i^2\right)^{\lambda}} \right) \right\rangle \tag{30}$$

Based on Definition 7, we can derive the following properties easily.

Theorem 3. For any two spherical uncertain linguistic numbers $\tilde{a}_i = \left\langle \left[s_{\theta_i}, s_{\tau_i} \right], \left(P_i, I_i, N_i \right) \right\rangle$ and $\tilde{a}_j = \left\langle \left[s_{\theta_j}, s_{\tau_j} \right], \left(P_j, I_j, N_j \right) \right\rangle$ and $\lambda \geq 0$, it can be proved the calculation rules shown as follows

$$\tilde{a}_i \oplus \tilde{a}_j = \tilde{a}_j \oplus \tilde{a}_i; \tag{31}$$

$$\tilde{a}_i \otimes \tilde{a}_j = \tilde{a}_j \otimes \tilde{a}_i; \tag{32}$$

$$\lambda\left(\tilde{a}_i \oplus \tilde{a}_j\right) = \lambda \tilde{a}_i \oplus \lambda \tilde{a}_j; \tag{33}$$

$$\lambda_i \tilde{a}_i \oplus \lambda_j \tilde{a}_i = \left(\lambda_i \oplus \lambda_j\right) \tilde{a}_i, \lambda_i, \lambda_j \geq 0; \tag{34}$$

$$\left(\tilde{a}_i\right)^{\lambda_i} \otimes \left(\tilde{a}_i\right)^{\lambda_j} = \left(\tilde{a}_i\right)^{\lambda_i + \lambda_j}, \lambda_i, \lambda_j \geq 0; \tag{35}$$

$$\left(\tilde{a}_i\right)^{\lambda_i} \otimes \left(\tilde{a}_j\right)^{\lambda_i} = \left(\tilde{a}_i \otimes \tilde{a}_j\right)^{\lambda_i}, \lambda_i \geq 0; \tag{36}$$

$$\left(\left(\tilde{a}_i\right)^{\lambda_i}\right)^{\lambda_j} = \left(\tilde{a}_i\right)^{\lambda_i \lambda_j}, \lambda_i, \lambda_j \geq 0. \tag{37}$$

3. Spherical uncertain linguistic arithmetic aggregation operators

In this section, we shall develop some arithmetic aggregation operators with spherical uncertain linguistic information, such as spherical linguistic weighted averaging (SULWA) operator, spherical uncertain linguistic ordered weighted averaging (SULOWA) operator, spherical uncertain linguistic hybrid averaging (SULHA) operator, and generalized spherical uncertain linguistic weighted averaging (GSULWA) operator.

Definition 8. (SULWA) Let $\tilde{a}_i = \left\langle \left[s_{\theta_i}, s_{\tau_i} \right], \left(P_i, I_i, N_i \right) \right\rangle$ (i=1, 2, …, n)be any collection of SULNs and SULWA: SULNn→SULN, then SULWA can be described as

$$SULWA_w\left(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n\right) = \overset{n}{\underset{i=1}{\oplus}} \left(w_i \tilde{a}_i\right) \tag{38}$$

Where $w = \left(w_1, w_2, \ldots, w_n\right)^T$ be the weight vector of $\tilde{a}_i$ (i=1, 2, …, n), and $w_i > 0, \sum_{i=1}^{n} w_i = 1$.

Theorem 4. Let $\tilde{a}_i = \left\langle \left[s_{\theta_i}, s_{\tau_i} \right], \left(P_i, I_i, N_i \right) \right\rangle$ (i=1, 2, …, n) be any collection of SULNs. Then, by utilizing Definition 8 and the operational properties of SULNs form Theorem 3, we can obtain the following result.

$$\begin{aligned} & SULWA_w\left(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n\right) = \overset{n}{\underset{i=1}{\oplus}} \left(w_i \tilde{a}_i\right) \\ & = \left\langle \left[s_{\sum_{i=1}^{n} w_i \theta_i}, s_{\sum_{i}^{n} w_i \tau_i} \right], \left(\sqrt{1 - \prod_{i=1}^{n} \left(1 - \left(P_i\right)^2\right)^{w_i}}, \prod_{i=1}^{n} \left(I_i\right)^{w_i}, \prod_{i=1}^{n} \left(N_i\right)^{w_i} \right) \right\rangle \end{aligned} \tag{39}$$

Where $w = \left(w_1, w_2, \ldots, w_n\right)^T$ be the weight vector of $\tilde{a}_i$ (i=1, 2,…, n), and $w_i > 0, \sum_{i=1}^{n} w_i = 1$.

Proof: We prove Eq. (39) by mathematical induction on n.

When n = 2, we have

$$SULWA_w(\tilde{a}_1,\tilde{a}_2) = w_1\tilde{a}_1 \oplus w_2\tilde{a}_2$$

From Definition 7 and Theorem 3, we can see that both $w_1\tilde{a}_1$ and $w_2\tilde{a}_2$ are SULNs, and the value of $w_1\tilde{a}_1 \oplus w_2\tilde{a}_2$ is also a SULN. From the operational laws of spherical uncertain linguistic number, we have

$$w_1\tilde{a}_1 = \left\langle \left[w_1 s_{\theta_1}, w_1 s_{\tau_1}\right], \left(\sqrt{1-\left(1-P_1^2\right)^{w_1}}, I_1^{w_1}, N_1^{w_1}\right)\right\rangle$$

$$w_2\tilde{a}_2 = \left\langle \left[w_2 s_{\theta_2}, w_2 s_{\tau_2}\right], \left(\sqrt{1-\left(1-P_2^2\right)^{w_2}}, I_2^{w_2}, N_2^{w_2}\right)\right\rangle$$

Then

$$SULWA_w(\tilde{a}_1,\tilde{a}_2) = w_1\tilde{a}_1 \oplus w_2\tilde{a}_2$$

$$= \left\langle \left[w_1 s_{\theta_1} + w_2 s_{\theta_2}, w_1 s_{\tau_1} + w_2 s_{\tau_2}\right], \left(\begin{array}{l}\sqrt{\left(1-\left(1-P_1^2\right)^{w_1}\right)+\left(1-\left(1-P_2^2\right)^{w_2}\right)-\left(1-\left(1-P_1^2\right)^{w_1}\right)\left(1-\left(1-P_2^2\right)^{w_2}\right)},\\ I_1^{w_1}\cdot I_2^{w_2},\\ N_1^{w_1}\cdot N_2^{w_2}\end{array}\right)\right\rangle$$

$$= \left\langle \left[s_{w_1\theta_1+w_2\theta_2}, s_{w_1\tau_1+w_2\tau_2}\right], \left(\sqrt{1-\left(1-P_1^2\right)^{w_1}\left(1-P_2^2\right)^{w_2}}, I_1^{w_1}\cdot I_2^{w_2}, N_1^{w_1}\cdot N_2^{w_2}\right)\right\rangle$$

Suppose that n = k, Eq.(7) holds, i.e.,

$$SULWA_w(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_k) = w_1\tilde{a}_1 \oplus w_2\tilde{a}_2 \oplus \ldots \oplus w_k\tilde{a}_k$$

$$= \left\langle \left[s_{\sum_{i=1}^{k} w_i\theta_i}, s_{\sum_{i}^{k} w_i\tau_i}\right], \left(\sqrt{1-\prod_{i=1}^{k}\left(1-(P_i)^2\right)^{w_i}}, \prod_{i=1}^{k}(I_i)^{w_i}, \prod_{i=1}^{k}(N_i)^{w_i}\right)\right\rangle$$

And the aggregated value is a SULN, then when n=k+1, by the operational laws of SULN, we have

$$SULWA_w(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_{k+1}) = w_1\tilde{a}_1 \oplus w_2\tilde{a}_2 \oplus \ldots \oplus w_k\tilde{a}_k \oplus w_{k+1}\tilde{a}_{k+1}$$

$$= \left\langle \left[s_{\sum_{i=1}^{k} w_i\theta_i} + s_{w_{k+1}\theta_{k+1}}, s_{\sum_{i}^{k} w_i\tau_i} + s_{w_{k+1}\tau_{k+1}}\right], \left(\begin{array}{l}\sqrt{\left(1-\prod_{i=1}^{k}\left(1-(P_i)^2\right)^{w_i}\right)+\left(1-\left(1-P_{k+1}^2\right)^{w_{k+1}}\right)-\left(1-\prod_{i=1}^{k}\left(1-(P_i)^2\right)^{w_i}\right)\left(1-\left(1-P_{k+1}^2\right)^{w_{k+1}}\right)},\\ \prod_{i=1}^{k}(I_i)^{w_i}\cdot(I_{k+1})^{w_{k+1}},\\ \prod_{i=1}^{k}(N_i)^{w_i}\cdot(N_{k+1})^{w_{k+1}}\end{array}\right)\right\rangle$$

$$= \left\langle \left[s_{\sum_{i=1}^{k+1} w_i\theta_i}, s_{\sum_{i}^{k+1} w_i\tau_i}\right], \left(\sqrt{\left(1-\prod_{i=1}^{k+1}\left(1-(P_i)^2\right)^{w_i}\right)}, \prod_{i=1}^{k+1}(I_i)^{w_i}, \prod_{i=1}^{k+1}(N_i)^{w_i}\right)\right\rangle$$

By which aggregated value is also a SULN, therefore, when n = k+1, Eq. (39) holds.

Thus, by Eq. (31) and Eq. (32), we know that Eq.(39) holds for all n. the proof is completed.

It can be easily proved that the SULWA operator has the following properties.

Theorem 5. (Idempotency) If all $\tilde{a}_i (i=1,2,\ldots,n)$ are equal, i.e. $\tilde{a}_i = \tilde{a}$ for all i, then

$$SULWA_w(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n) = SULWA_w(\tilde{a},\tilde{a},\ldots,\tilde{a}) = \tilde{a}$$

Theorem 6. (Boundedness) Let $\tilde{a}_i (i=1,2,\ldots,n)$ be a collection of SULNs, and let $\tilde{a}^- = \min_i \tilde{a}_i, \tilde{a}^+ = \max_i \tilde{a}_i$, then

$$\tilde{a}^- \le SULWA_w(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n) \le \tilde{a}^+$$

Theorem 7. (Monotonicity) Let $\tilde{a}_i (i=1,2,\ldots,n)$ and $\tilde{a}'_i (i=1,2,\ldots,n)$ be two set of SULNs, if $\tilde{a}_i \le \tilde{a}'_i$, for all I, then

$$SULWA_w\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n\right) \le SULWA_w\left(\tilde{a}'_1,\tilde{a}'_2,\ldots,\tilde{a}'_n\right)$$

Further, we give a spherical uncertain linguistic ordered weighted averaging (SULOWA) operator below:

Definition 9. (SULOWA) Let $\tilde{a}_i = \left\langle \left[s_{\theta_i}, s_{\tau_i} \right], \left(P_i, I_i, N_i\right) \right\rangle$ (i=1, 2, …, n)be any collection of SULNs and SULOWA: SULNn → SULN, then SULOWA can be described as

$$SULOWA_w\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n\right) = \overset{n}{\underset{i=1}{\oplus}}\left(w_i \tilde{a}_{\eta(i)}\right) \tag{40}$$

With dimensions n, where the i-th biggest weighted value is $\tilde{a}_{\eta(i)}$ consequently by total order $\tilde{a}_{\eta(1)} \ge \tilde{a}_{\eta(2)} \ge \cdots \ge \tilde{a}_{\eta(n)}$. $w=\left(w_1,w_2,\ldots,w_n\right)^T$ is the associated weight vector such that $w_i > 0$ and $\sum_{i=1}^{n} w_i = 1$ (i=1, 2, …, n).

Theorem 8. Let Let $\tilde{a}_i = \left\langle \left[s_{\theta_i}, s_{\tau_i} \right], \left(P_i, I_i, N_i\right) \right\rangle$ (i=1, 2, …, n) be any collection of SULNs. Then by utilizing Definition 9 and the operational properties of SULNs from Theorem 3, we can get the following outcome.

$$SULOWA_w\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n\right) = \overset{n}{\underset{i=1}{\oplus}}\left(w_i \tilde{a}_{\eta(i)}\right)$$
$$= \left\langle \left[s_{\sum_{i=1}^{n} w_i \theta_{\eta(i)}}, s_{\sum_{i}^{n} w_i \tau_{\eta(i)}} \right], \left(\sqrt{1-\prod_{i=1}^{n}\left(1-\left(P_{\eta(i)}\right)^2\right)^{w_i}}, \prod_{i=1}^{n}\left(I_{\eta(i)}\right)^{w_i}, \prod_{i=1}^{n}\left(N_{\eta(i)}\right)^{w_i} \right) \right\rangle \tag{41}$$

Where $w=\left(w_1,w_2,\ldots,w_n\right)^T$ be the associated weight vector of SULNs such that $w_i \ge 0$ and $\sum_{i=1}^{n} w_i = 1$ (i=1, 2, …, n).

It can be easily proved that SULOWA operator has the following properties.

Theorem 9. (Idempotency) If all $\tilde{a}_i (i=1,2,\ldots,n)$ are equal, i.e. $\tilde{a}_i = \tilde{a}$ for all i, then

$$SULOWA_w\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n\right) = SULOWA_w\left(\tilde{a},\tilde{a},\ldots,\tilde{a}\right) = \tilde{a}$$

Theorem 10. (Boundedness) Let $\tilde{a}_i (i=1,2,\ldots,n)$ be a collection of SULNs, and let $\tilde{a}^- = \min_i \tilde{a}_i, \tilde{a}^+ = \max_i \tilde{a}_i$, then

$$\tilde{a}^- \le SULOWA_w\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n\right) \le \tilde{a}^+$$

Theorem 11. (Monotonicity) Let $\tilde{a}_i (i=1,2,\ldots,n)$ and $\tilde{a}'_i (i=1,2,\ldots,n)$ be two set SULNs, if $\tilde{a}_i \le \tilde{a}'_i$, for all i, then

$$SULOWA_w\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n\right) \le SULOWA_w\left(\tilde{a}'_1,\tilde{a}'_2,\ldots,\tilde{a}'_n\right)$$

Theorem 12. (Commutativity) Let $\tilde{a}_i (i=1,2,\ldots,n)$ and $\tilde{a}_{\eta(i)} (i=1,2,\ldots,n)$ be two set SULNs, for all i, then

$$SULOWA_w\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n\right) = SULOWA_w\left(\tilde{a}_{\eta(1)},\tilde{a}_{\eta(2)},\ldots,\tilde{a}_{\eta(n)}\right)$$

Where $\tilde{a}_{\eta(i)} (i=1,2,\ldots,n)$ is any permutation of $\tilde{a}_i (i=1,2,\ldots,n)$.

From Definition 8-9, we know that the SULWA operator only weights the spherical uncertain linguistic number itself, while that SULOWA operator weights the ordered positions of the spherical uncertain linguistic number instead of weighting the arguments itself. Therefore, the weights represent two different aspects in both the SULWA and SULOWA operators. However, both the operators consider only one of them. To solve this drawback, in the following we shall propose the spherical uncertain linguistic hybrid average (SULHA) operator.

Definition 10. (SULHA) Let $\tilde{a}_i = \left\langle \left[s_{\theta_i}, s_{\tau_i} \right], \left(P_i, I_i, N_i \right) \right\rangle$ (i=1, 2, …, n)be any collection of SULNs and SULHA: SULNn → SULN, then SULHA can be described as

$$SULHA_{\mu,w}\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n\right) = \overset{n}{\underset{i=1}{\oplus}}\left(\mu_i \dot{\tilde{a}}_{\eta(i)}\right)$$
$$= \left\langle \left[\dot{s}_{\sum_{i=1}^{n}\mu_i\theta_{\eta(i)}}, \dot{s}_{\sum_{i}^{n}\mu_i\tau_{\eta(i)}} \right], \left(\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\dot{P}_{\eta(i)}\right)^2\right)^{\mu_i}}, \prod_{i=1}^{n}\left(\dot{I}_{\eta(i)}\right)^{\mu_i}, \prod_{i=1}^{n}\left(\dot{N}_{\eta(i)}\right)^{\mu_i} \right) \right\rangle \tag{42}$$

Where $\mu = \left(\mu_1,\mu_2,\ldots,\mu_n\right)^T$ is the associated weighting vector, with $\mu_i \in [0,1]$, $\sum_{i=1}^{n}\mu_i = 1$, and $\dot{\tilde{a}}_{\eta(i)}$ is the i-th largest element of the spherical uncertain linguistic arguments $\dot{\tilde{a}}_i\left(\dot{\tilde{a}}_i = \left(nw_i\right)\tilde{a}_i, i = 1,2,\ldots,n\right)$, $w = \left(w_1,w_2,\ldots,w_n\right)^T$ is the weighting vector of spherical uncertain linguistic arguments $\tilde{a}_i (i = 1,2,\ldots,n)$, with $w_i \in [0,1], \sum_{i=1}^{n} w_i = 1$, and n is the balancing coefficient. Especially, if $\mu = \left(1/n, 1/n, \ldots, 1/n\right)^T$, the SULHA is reduced to the spherical uncertain linguistic weighted average (SULWA), if $w = \left(1/n, 1/n, \ldots, 1/n\right)^T$, then SULHA is reduced to the spherical uncertain linguistic ordered weighted average (SULOWA) operator.

Definition 11. (GSULWA) Let $\tilde{a}_i = \left\langle \left[s_{\theta_i}, s_{\tau_i} \right], \left(P_i, I_i, N_i \right) \right\rangle$ (i=1, 2, …, n) be any collection of SULNs and GSULWA: SULNn → SULN, then GSULWA can be described as

$$GSULWA\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n\right) = \left(\overset{n}{\underset{i=1}{\oplus}} w_i \tilde{a}_i^{\lambda}\right)^{1/\lambda}$$
$$= \left\langle \left[s_{\left(\sum_{i=1}^{n} w_i\left(\theta_i\right)^{\lambda}\right)^{1/\lambda}}, s_{\left(\sum_{i=1}^{n} w_i\left(\tau_i\right)^{\lambda}\right)^{1/\lambda}} \right], \left(\left(1-\prod_{i=1}^{n}\left(1-\left(P_i\right)^{2\lambda}\right)^{w_i}\right)^{1/2\lambda}, \prod_{i=1}^{n}\left(I_i\right)^{w_i}, \sqrt{1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\left(N_i\right)^2\right)^{\lambda}\right)^{w_i}\right)^{1/\lambda}} \right) \right\rangle \tag{43}$$

Where $w = \left(w_1,w_2,\ldots,w_n\right)^T$ be the weight vector of SULNs, with $w_i \in [0,1], \sum_{i=1}^{n} w_i = 1$, and $\lambda > 0$. Especially, if $\lambda = 1$, the GSULWA is reduced to the spherical uncertain linguistic weighted average (SULWA).

4. Spherical uncertain linguistic geometric aggregation operators

This section describes some geometric aggregation operators with spherical uncertain linguistic information, such as spherical uncertain linguistic weighted geometric (SULWG) operator, spherical uncertain linguistic ordered weighted geometric (SULOWG) operator, spherical uncertain linguistic hybrid geometric (SULHG) operator and generalized spherical uncertain linguistic weighted geometric (GSULWG) operator.

Definition 12. (SULWG) Let $\tilde{a}_i = \left\langle \left[s_{\theta_i}, s_{\tau_i} \right], \left(P_i, I_i, N_i \right) \right\rangle$ (i=1, 2, …, n) be any collection of SULNs and SULWG: SULNn → SULN, then SULWG can be described as

$$SULWG_w\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n\right) = \bigotimes_{i=1}^{n}\left(\tilde{a}_i\right)^{w_i} \tag{44}$$

Where $w = \left(w_1,w_2,\ldots,w_n\right)^T$ be the weight vector of SULNs, with $w_i \in [0,1], \sum_{i=1}^{n} w_i = 1$

Based on Definition 12 and Theorem 3, we can obtain the following result.

Theorem 13. The aggregated value by using SULWG operator is also a SULN, where

$$SULWG_w\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n\right)=\bigotimes_{i=1}^{n}\left(\tilde{a}_i\right)^{w_i}$$
$$=\left\langle\left[s_{\prod_{i=1}^{n}(\theta_i)^{w_i}},s_{\prod_{i=1}^{n}(\tau_i)^{w_i}}\right],\left(\prod_{i=1}^{n}\left(P_i\right)^{w_i},\prod_{i=1}^{n}\left(I_i\right)^{w_i},\sqrt{1-\prod_{i=1}^{n}\left(1-\left(N_i\right)^2\right)^{w_i}}\right)\right\rangle \quad (45)$$

Where $w=\left(w_1,w_2,\ldots,w_n\right)^T$ be the weight vector of SULNs, with $w_i\in[0,1],\sum_{i=1}^{n}w_i=1$

Proof: We prove Eq. (45) by mathematical induction on n.

When n = 2, we have

$$SULWG_w\left(\tilde{a}_1,\tilde{a}_2\right)=\left(\tilde{a}_1\right)^{w_1}\otimes\left(\tilde{a}_2\right)^{w_2}$$

By Definition 7 and Theorem 3, we can see that both $\left(\tilde{a}_1\right)^{w_1}$ and $\left(\tilde{a}_2\right)^{w_2}$ are SULNs, and the value of $\left(\tilde{a}_1\right)^{w_1}\otimes\left(\tilde{a}_2\right)^{w_2}$ is also a SULN. From the operational laws of spherical uncertain linguistic number, we have

$$\left(\tilde{a}_1\right)^{w_1}=\left\langle\left[s_{(\theta_1)^{w_1}},s_{(\tau_1)^{w_1}}\right],\left(P_1^{w_1},I_1^{w_1},\sqrt{1-\left(1-N_1^2\right)^{w_1}}\right)\right\rangle$$
$$\left(\tilde{a}_2\right)^{w_2}=\left\langle\left[s_{(\theta_2)^{w_2}},s_{(\tau_2)^{w_2}}\right],\left(P_2^{w_2},I_2^{w_2},\sqrt{1-\left(1-N_2^2\right)^{w_2}}\right)\right\rangle$$

Then

$$SULWG_w\left(\tilde{a}_1,\tilde{a}_2\right)=\left(\tilde{a}_1\right)^{w_1}\otimes\left(\tilde{a}_2\right)^{w_2}$$
$$=\left\langle\left[s_{(\theta_1)^{w_1}}\times s_{(\theta_2)^{w_2}},s_{(\tau_1)^{w_1}}\times s_{(\tau_2)^{w_2}}\right],\left(\begin{array}{l}N_1^{w_1}\cdot N_2^{w_2},\\ I_1^{w_1}\cdot I_2^{w_2},\\ \sqrt{\left(1-\left(1-P_1^2\right)^{w_1}\right)+\left(1-\left(1-P_2^2\right)^{w_2}\right)-\left(1-\left(1-P_1^2\right)^{w_1}\right)\left(1-\left(1-P_2^2\right)^{w_2}\right)}\end{array}\right)\right\rangle$$
$$=\left\langle\left[s_{(\theta_1)^{w_1}(\theta_2)^{w_2}},s_{(\tau_1)^{w_1}(\tau_2)^{w_2}}\right],\left(\sqrt{1-\left(1-P_1^2\right)^{w_1}\left(1-P_2^2\right)^{w_2}},I_1^{w_1}\cdot I_2^{w_2},N_1^{w_1}\cdot N_2^{w_2}\right)\right\rangle$$

Suppose that n = k, Eq. (45) holds, i.e.,

$$SULWG_w\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_k\right)=\bigotimes_{i=1}^{k}\left(\tilde{a}_i\right)^{w_i}$$
$$=\left\langle\left[s_{\prod_{i=1}^{k}(\theta_i)^{w_i}},s_{\prod_{i=1}^{k}(\tau_i)^{w_i}}\right],\left(\prod_{i=1}^{k}\left(P_i\right)^{w_i},\prod_{i=1}^{k}\left(I_i\right)^{w_i},\sqrt{1-\prod_{i=1}^{k}\left(1-\left(N_i\right)^2\right)^{w_i}}\right)\right\rangle$$

And the aggregated value is a SULN, then when n=k+1, by the operational laws of SULN, we have

$$ULWG_w\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_{k+1}\right)=w_1\tilde{a}_1\otimes w_2\tilde{a}_2\otimes\ldots\otimes w_k\tilde{a}_k\otimes w_{k+1}\tilde{a}_{k+1}$$
$$=\left\langle\left[s_{\prod_{i=1}^{k}(\theta_i)^{w_i}}\times s_{(\theta_{k+1})^{w_{k+1}}},s_{\prod_{i=1}^{k}(\tau_i)^{w_i}}\times s_{(\tau_{k+1})^{w_{k+1}}}\right],\left(\begin{array}{l}\prod_{i=1}^{k}\left(P_i\right)^{w_i}\cdot\left(P_{k+1}\right)^{w_{k+1}},\\ \prod_{i=1}^{k}\left(I_i\right)^{w_i}\cdot\left(I_{k+1}\right)^{w_{k+1}},\\ \sqrt{\left(1-\prod_{i=1}^{k}\left(1-\left(N_i\right)^2\right)^{w_i}\right)+\left(1-\left(1-N_{k+1}^2\right)^{w_{k+1}}\right)-\left(1-\prod_{i=1}^{k}\left(1-\left(N_i\right)^2\right)^{w_i}\right)\left(1-\left(1-N_{k+1}^2\right)^{w_{k+1}}\right)}\end{array}\right)\right\rangle$$
$$=\left\langle\left[s_{\prod_{i=1}^{k+1}(\theta_i)^{w_i}},s_{\prod_{i=1}^{k+1}(\tau_i)^{w_i}}\right],\left(\prod_{i=1}^{k+1}\left(P_i\right)^{w_i},\prod_{i=1}^{k+1}\left(I_i\right)^{w_i},\sqrt{\left(1-\prod_{i=1}^{k+1}\left(1-\left(N_i\right)^2\right)^{w_i}\right)}\right)\right\rangle$$

By which aggregated value is also a SULN, therefore, when n=k+1, Eq. (45) holds.

Thus, by (31) and (32), we know that Eq. (45) holds for all n. the proof is completed.

It can be easily proved that the SULWG operator has the following properties.

Theorem 14. (Idempotency) If all $\tilde{a}_i (i=1,2,\ldots,n)$ are equal, i.e. $\tilde{a}_i = \tilde{a}$ for all i, then

$$SULWG_w(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = SULWG_w(\tilde{a}, \tilde{a}, \ldots, \tilde{a}) = \tilde{a}$$

Theorem 15 (Boundedness) Let $\tilde{a}_i (i=1,2,\ldots,n)$ be a collection of SULNs, and let $\tilde{a}^- = \min_i \tilde{a}_i, \tilde{a}^+ = \max_i \tilde{a}_i$, then

$$\tilde{a}^- \le SULWG_w(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \le \tilde{a}^+$$

Theorem 16. (Monotonicity) Let $\tilde{a}_i (i=1,2,\ldots,n)$ and $\tilde{a}'_i (i=1,2,\ldots,n)$ be two set of SULNs, if $\tilde{a}_i \le \tilde{a}'_i$, for all i, then

$$SULWG_w(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \le SULWG_w(\tilde{a}'_1, \tilde{a}'_2, \ldots, \tilde{a}'_n)$$

Further, we give a spherical uncertain linguistic ordered weighted geometric (SULOWG) operator below:

Definition 13. (SULOWG) Let $\tilde{a}_i = \left\langle \left[s_{\theta_i}, s_{\tau_i} \right], \left(P_i, I_i, N_i \right) \right\rangle$ (i=1, 2, …, n) be any collection of SULNs and SULOWG: SULNn → SULN, then SULOWG can be described as

$$SULOWG_w(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \bigotimes_{i=1}^{n} \left(\tilde{a}_{\eta(i)} \right)^{w_i} \tag{46}$$

With dimensions n, where the i-th biggest weighted value is $\tilde{a}_{\eta(i)}$ consequently by total order $\tilde{a}_{\eta(1)} \ge \tilde{a}_{\eta(2)} \ge \cdots \ge \tilde{a}_{\eta(n)}$. $w = (w_1, w_2, \ldots, w_n)^T$ is the associated weight vector of SULN, with $w_i > 0$ and $\sum_{i=1}^{n} w_i = 1$ (i=1, 2, …, n).

Theorem 17. Let $\tilde{a}_i = \left\langle \left[s_{\theta_i}, s_{\tau_i} \right], \left(P_i, I_i, N_i \right) \right\rangle$ (i=1, 2, …, n) be any collection of SULNs. Then, by utilizing Definition 13 and the operational properties of SULNs from Theorem 3, we can obtain the following outcome.

$$SULOWG_w(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = \bigotimes_{i=1}^{n} \left(\tilde{a}_{\eta(i)} \right)^{w_i}$$

$$= \left\langle \left[s_{\left(\prod_{i=1}^{n} \theta_{\eta(i)} \right)^{w_i}}, s_{\left(\prod_{i=1}^{n} \tau_{\eta(i)} \right)^{w_i}} \right], \left(\prod_{i=1}^{n} \left(P_{\eta(i)} \right)^{w_i}, \prod_{i=1}^{n} \left(I_{\eta(i)} \right)^{w_i}, \sqrt{1 - \prod_{i=1}^{n} \left(1 - \left(N_{\eta(i)} \right)^2 \right)^{w_i}} \right) \right\rangle \tag{47}$$

Where $w = (w_1, w_2, \ldots, w_n)^T$ be the associated weight vector of SULNs such that $w_i \ge 0$ and $\sum_{i=1}^{n} w_i = 1$ (i=1, 2, …, n).

It can be easily proved that the SULOWG operator has the following properties.

Theorem 18. (Idempotency) If all $\tilde{a}_i (i=1,2,\ldots,n)$ are equal, i.e. $\tilde{a}_i = \tilde{a}$ for all i, then

$$SULOWG_w(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) = SULOWG_w(\tilde{a}, \tilde{a}, \ldots, \tilde{a}) = \tilde{a}$$

Theorem 19. (Boundedness) Let $\tilde{a}_i (i=1,2,\ldots,n)$ be a collection of SULNs, and let $\tilde{a}^- = \min_i \tilde{a}_i, \tilde{a}^+ = \max_i \tilde{a}_i$, then

$$\tilde{a}^- \le SULOWG_w(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \le \tilde{a}^+$$

Theorem 20. (Monotonicity) Let $\tilde{a}_i (i=1,2,\ldots,n)$ and $\tilde{a}'_i (i=1,2,\ldots,n)$ be two set of SULNs, if $\tilde{a}_i \le \tilde{a}'_i$, for all i, then

$$SULOWG_w(\tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_n) \le SULOWG_w(\tilde{a}'_1, \tilde{a}'_2, \ldots, \tilde{a}'_n)$$

Theorem 21. (Commutativity) Let $\tilde{a}_i (i=1,2,\ldots,n)$ and $\tilde{a}_{\eta(i)} (i=1,2,\ldots,n)$ be two set SULNs, for all i, then

$$SULOWG_w\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n\right)=SULOWG_w\left(\tilde{a}_{\eta(1)},\tilde{a}_{\eta(2)},\ldots,\tilde{a}_{\eta(n)}\right)$$

Where $\tilde{a}_{\eta(i)} (i=1,2,\ldots,n)$ is any permutation of $\tilde{a}_i (i=1,2,\ldots,n)$.

From Definitions 12-13, we know that the SULWG operator only weights that spherical uncertain linguistic number itself, which the SULOWG operator weights the ordered positions of the spherical uncertain linguistic number instead of weighting the arguments itself. Therefore, the weights represent two different aspects in both the SULWG and SULOWG operators. However, both the operators consider only one of them. To solve this drawback, in the following we shall propose the spherical uncertain linguistic hybrid geometric (SULHG) operator.

Definition 14. (SULHG) Let $\tilde{a}_i=\left\langle\left[s_{\theta_i},s_{\tau_i}\right],\left(P_i,I_i,N_i\right)\right\rangle$ (i=1, 2, …, n) be any collection of SULNs and SULHG: SULNn → SULN, then SULHG can be described as

$$SULHG_{\mu,w}\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n\right)=\bigotimes_{i=1}^{n}\left(\dot{\tilde{a}}_{\eta(i)}\right)^{\mu_i}$$
$$\left\langle\left[\dot{s}_{\prod_{i=1}^{n}\left(\theta_{\eta(i)}\right)^{\mu_i}},\dot{s}_{\prod_{i=1}^{n}\left(\tau_{\eta(i)}\right)^{\mu_i}}\right],\left(\prod_{i=1}^{n}\left(\dot{P}_{\eta(i)}\right)^{\mu_i},\prod_{i=1}^{n}\left(\dot{I}_{\eta(i)}\right)^{\mu_i},\sqrt{1-\prod_{i=1}^{n}\left(1-\left(\dot{N}_{\eta(i)}\right)^2\right)^{\mu_i}}\right)\right\rangle \quad (48)$$

Where $\mu=\left(\mu_1,\mu_2,\ldots,\mu_n\right)^T$ is the associated weighting vector, with $\mu_i\in[0,1]$, $\sum_{i=1}^{n}\mu_i=1$, and $\dot{\tilde{a}}_{\eta(i)}$ is the i-th largest element of the spherical uncertain linguistic arguments $\dot{\tilde{a}}_i\left(\dot{\tilde{a}}_i=\left(\tilde{a}_i\right)^{nw_i},i=1,2,\ldots,n\right)$, $w=\left(w_1,w_2,\ldots,w_n\right)^T$ is the weighting vector of spherical uncertain linguistic arguments $\tilde{a}_i (i=1,2,\ldots,n)$, with $w_i\in[0,1],\sum_{i=1}^{n}w_i=1$, and n is the balancing coefficient. Especially, if $\mu=\left(1/n,1/n,\ldots,1/n\right)^T$, the SULHG is reduced to the spherical uncertain linguistic weighted geometric (SULWG), if $w=\left(1/n,1/n,\ldots,1/n\right)^T$, then SULHG is reduced to the spherical uncertain linguistic ordered weighted geometric (SULOWG) operator.

Definition 15. (GSULWG) Let $\tilde{a}_i=\left\langle\left[s_{\theta_i},s_{\tau_i}\right],\left(P_i,I_i,N_i\right)\right\rangle$ (i=1, 2, …, n) be any collection of SULNs and GSULWG: SULNn → SULN, then GSULWG can be described as

$$GSULWG_w\left(\tilde{a}_1,\tilde{a}_2,\ldots,\tilde{a}_n\right)=\frac{1}{\lambda}\bigotimes_{i=1}^{n}\left(\lambda\tilde{a}_i\right)^{w_i}$$
$$=\left\langle\left[s_{\frac{1}{\lambda}\prod_{i=1}^{n}\left(\lambda\theta_i\right)^{w_i}},s_{\frac{1}{\lambda}\prod_{i=1}^{n}\left(\lambda\tau_i\right)^{w_i}}\right],\left(\sqrt{1-\left(1-\prod_{i=1}^{n}\left(1-\left(1-\left(P_i\right)^2\right)^{\lambda}\right)^{w_i}\right)^{1/\lambda}},\prod_{i=1}^{n}\left(I_i\right)^{w_i},\left(1-\prod_{i=1}^{n}\left(1-\left(N_i\right)^{2\lambda}\right)^{w_i}\right)^{1/2\lambda}\right)\right\rangle \quad (49)$$

Where $w=\left(w_1,w_2,\ldots,w_n\right)^T$ be the weight vector of SULNs, with $w_i\in[0,1],\sum_{i=1}^{n}w_i=1$, and $\lambda>0$. Especially, if $\lambda=1$, the GSULWG is reduced to the spherical uncertain linguistic weighted geometric (SULWG).

5. MADM method utilizing spherical uncertain linguistic aggregation operators

Consider a multi attribute decision making problem with spherical uncertain linguistic information. Let $A=\left\{A_1,A_2,\ldots,A_m\right\}$ be a discrete set of alternatives and $C=\left\{C_1,C_2,\ldots,C_n\right\}$ be the set of attributes. $w=\left(w_1,w_2,\ldots,w_n\right)^T$ is the weighting vector of the attribute $C_j\left(j=1,2,\ldots,n\right)$, where

$w_j \geq 0, j=1,2,\ldots,n$, $\sum_{j=1}^{n} w_j = 1$. Suppose that $\tilde{D}=\left(\tilde{d}_{ij}\right)_{m\times n}=\left\langle\left[s_{\theta_{ij}},s_{\tau_{ij}}\right],\left(P_{ij},I_{ij},N_{ij}\right)\right\rangle_{m\times n}$ is the spherical uncertain linguistic decision matrix, where $\tilde{d}_{ij}$ takes the form of the spherical uncertain linguistic variable.

In the following, we apply the SULHA or SULHG or GSULWA or GSULWG operator to solve this MADM problem. The method involves the following steps:

Step 1. We utilize the decision information given in matrix $\tilde{D}$ and the SULHA operator, GSULWA operator, SULHG operator and GSULWG operator to derive the overall preference values $\tilde{p}_i$ (i=1,2,⋯,m) of the alternative A_i.

By Eq. (42) and Eq. (48), we can utilize the SULHA operator and SULHG operator

$$\tilde{p}_i = SULHA_{\mu,w}\left(\tilde{d}_{i1},\tilde{d}_{i2},\ldots,\tilde{d}_{in}\right)=\overset{n}{\underset{j=1}{\oplus}}\left(\mu_j\dot{\tilde{d}}_{i\rho(j)}\right)$$
$$=\left\langle\left[\dot{s}_{\sum_{j=1}^{n}\mu_j\theta_{i\rho(j)}},\dot{s}_{\sum_{j=1}^{n}\mu_j\tau_{i\rho(j)}}\right],\left(\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\dot{P}_{i\rho(j)}\right)^2\right)^{\mu_j}},\prod_{j=1}^{n}\left(\dot{I}_{i\rho(j)}\right)^{\mu_j},\prod_{j=1}^{n}\left(\dot{N}_{i\rho(j)}\right)^{\mu_j}\right)\right\rangle \quad (50)$$

$$\tilde{p}_i = SULHG_{\mu,w}\left(\tilde{d}_{i1},\tilde{d}_{i2},\ldots,\tilde{d}_{in}\right)=\overset{n}{\underset{j=1}{\otimes}}\left(\dot{\tilde{d}}_{i\rho(j)}\right)^{\mu_j}$$
$$=\left\langle\left[\dot{s}_{\prod_{j=1}^{n}\left(\theta_{i\rho(j)}\right)^{\mu_j}},\dot{s}_{\prod_{j=1}^{n}\left(\tau_{i\rho(j)}\right)^{\mu_j}}\right],\left(\prod_{j=1}^{n}\left(\dot{P}_{i\rho(j)}\right)^{\mu_j},\prod_{j=1}^{n}\left(\dot{I}_{i\rho(j)}\right)^{\mu_j},\sqrt{1-\prod_{j=1}^{n}\left(1-\left(\dot{N}_{i\rho(j)}\right)^2\right)^{\mu_j}}\right)\right\rangle \quad (51)$$

Where $\mu=\left(\mu_1,\mu_2,\ldots,\mu_n\right)^T$ is the associated weighting vector, with $\mu_j\in[0,1]$, $\sum_{j=1}^{n}\mu_j=1$, and $\dot{\tilde{d}}_{i\rho(j)}$ is the j-th largest element of the spherical uncertain linguistic arguments $\dot{\tilde{d}}_j\left(\dot{\tilde{d}}_j=\left(\tilde{d}_j\right)^{nw_j}, j=1,2,\ldots,n\right)$, $w=\left(w_1,w_2,\ldots,w_n\right)^T$ is the weighting vector of spherical uncertain linguistic arguments $\tilde{d}_j\,(j=1,2,\ldots,n)$, with $w_j\in[0,1],\sum_{j=1}^{n}w_j=1$, and n is the balancing coefficient.

By Eq. (43) and Eq. (49), we can utilize the GSULWA operator and GSULWG operator

$$\tilde{p}_i = GSULWA\left(\tilde{d}_{i1},\tilde{d}_{i2},\ldots,\tilde{d}_{in}\right)=\left(\overset{n}{\underset{j=1}{\oplus}}w_j\tilde{d}_{ij}^{\lambda}\right)^{1/\lambda}$$
$$=\left\langle\left[s_{\left(\sum_{j=1}^{n}w_j\left(\theta_{ij}\right)^{\lambda}\right)^{1/\lambda}},s_{\left(\sum_{j=1}^{n}w_j\left(\tau_{ij}\right)^{\lambda}\right)^{1/\lambda}}\right],\left(\left(1-\prod_{j=1}^{n}\left(1-\left(P_{ij}\right)^{2\lambda}\right)^{w_j}\right)^{1/2\lambda},\prod_{j=1}^{n}\left(I_{ij}\right)^{w_j},\sqrt{1-\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(N_{ij}\right)^2\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}}\right)\right\rangle \quad (52)$$

$$\tilde{p}_i = GSULWG_w\left(\tilde{d}_{i1},\tilde{d}_{i2},\ldots,\tilde{d}_{in}\right)=\frac{1}{\lambda}\bigotimes_{j=1}^{n}\left(\lambda\tilde{d}_{ij}\right)^{w_j}$$
$$=\left\langle\left[s_{\frac{1}{\lambda}\prod_{j=1}^{n}\left(\lambda\theta_{ij}\right)^{w_j}},s_{\frac{1}{\lambda}\prod_{j=1}^{n}\left(\lambda\tau_{ij}\right)^{w_{ij}}}\right],\left(\sqrt{1-\left(1-\prod_{j=1}^{n}\left(1-\left(1-\left(P_{ij}\right)^2\right)^{\lambda}\right)^{w_j}\right)^{1/\lambda}},\prod_{j=1}^{n}\left(I_{ij}\right)^{w_j},\left(1-\prod_{j=1}^{n}\left(1-\left(N_{ij}\right)^{2\lambda}\right)^{w_j}\right)^{1/2\lambda}\right)\right\rangle \quad (53)$$

Where $w=\left(w_1,w_2,\ldots,w_n\right)^T$ be the weight vector of SULNs, with $w_j\in[0,1],\sum_{j=1}^{n}w_j=1$, and $\lambda>0$.

Step 2. By Eq. (25), we can calculate the expected value $EX(\tilde{p}_i)$ (i=1, 2,…, m) of the overall spherical uncertain linguistic numbers $\tilde{p}_i$ (i=1, 2,…, m). rank all of the alternatives $A_i\left(i=1,2,\ldots,m\right)$ and then select the best one(s). if there is no difference between two expected values $EX(\tilde{p}_i)$ and $EX(\tilde{p}_k)$, then by Eq. (26), we must calculate the accuracy value $AC(\tilde{p}_i)$ and $AC(\tilde{p}_k)$ of the collective overall preference values $\tilde{p}_i$ and $\tilde{p}_k$, respectively, and then rank the alternatives in accordance with the accuracy values.

Step 3. Rank all the alternatives A_i (i=1, 2,…, m) and select the best one(s) in accordance with $EX(\tilde{p}_i)$ and $AC(\tilde{p}_i)$ (i=1, 2,…,m).

Step 4. End.

6. Illustrative example

A shopping website wants to evaluate four Android smart phones (A_1, A_2, A_3, A_4 as alternatives) with different price performance and price performance at the same price to be launched, so as to explore the potential purchase intention of consumers. The website employs some experts to aid this decision-making, and evaluates these alternatives according to the following four attributes (suppose that the weight vector of the four attributes is w =(0.25, 0.23, 0.25, 0.27)T): (1) C_1 is fluency of system; (2) C_2 is UI design; (3) C_3 is functional diversity and (4) C_4 is hardware configuration. The four smart phones A_i (i=1, 2, 3, 4) are to be evaluated using the spherical uncertain linguistic numbers by the decision makers under the above four attributes, and construct the following spherical uncertain linguistic decision matrix $\tilde{D}=\left[\tilde{d}_{ij}\right]_{4\times 4}$ is shown in Table 1.

Table 1
the spherical uncertain linguistic decision matrix

	C1	C2	C3	C4
A1	<[s2,s3],(0.7,0.4,0.2)>	<[s3,s4],(0.6,0.4,0.3)>	<[s1,s2],(0.6,0.5,0.4) >	<[s6,s7],(0.5,0.2,0.3)>
A2	<[s1,s2],(0.7,0.5,0.3)>	<[s4,s5],(0.5,0.7,0.2)>	<[s2,s3],(0.7,0.3,0.3)>	<[s2,s3],(0.6,0.2,0.4)>
A3	<[s2,s4],(0.4,0.7,0.4)>	<[s2,s3],(0.7,0.3,0.2)>	<[s3,s4],(0.8,0.3,0.3)>	<[s4,s6],(0.6,0.3,0.3)>
A4	<[s4,s5],(0.8,0.3,0.4)>	<[s1,s2],(0.7,0.6,0.3)>	<[s3,s5],(0.6,0.4,0.3)>	<[s1,s3],(0.5,0.3,0.4)>

6.1 The evaluation steps

In the following, we apply the SULHA operator, SULHG operator, GSULWA operator and GSULWG operator to solve this MADM problem. The method involves the following steps:

Step 1. According to Table 1, firstly, aggregate all spherical uncertain linguistic numbers $\tilde{d}_{ij}\left(j=1,2,3,4\right)$ by using the SULHA/SULHG operator to derive the overall spherical uncertain linguistic numbers $\tilde{p}_i\left(i=1,2,3,4\right)$ of the smart phones A_i (i=1, 2, 3, 4), in which the associated weighting vector μ can be determined by using a regular increasing monotone quantifier Q [62], and get $\mu=\left(0.2,0.1,0.3,0.4\right)^T$. Secondly, utilize the GSULWA/GSULWG operator ($\lambda=1$) to aggregate all spherical uncertain linguistic numbers $\tilde{d}_{ij}\left(j=1,2,3,4\right)$, and get the overall spherical uncertain linguistic numbers $\tilde{p}_i\left(i=1,2,3,4\right)$ of the smart phones A_i (i=1, 2, 3, 4). The aggregating results are shown in Table 2.

Table 2
Aggregated results by the four operators

	SULHA	SULHG
A_1	<[s3.31,s4.50],(0.593,0.310,0.293)>	<[s2.76,s4.04],(0.565,0.310,0.323)>
A_2	<[s1.69,s2.96],(0.653,0.293,0.318)>	<[s1.89,s2.98],(0.30,0.293,0.344)>
A_3	<[s3.07,s4.78],(0.670,0.345,0.297)>	<[s3.10,s4.80],(0.604,0.345,0.319)>
A_4	<[s2.17,s3.89],(0.639,0.339,0.350)>	<[s1.84,s3.83],(0.589,0.339,0.370)>

Table 2
Continued

	GSULWA($\lambda=1$)	GSULWG($\lambda=1$)
A_1	<[s3.06,s4.06],(0.607,0.351,0.291)>	<[s2.48,s3.64],(0.594,0.351,0.310)>
A_2	<[s2.21,s3.21],(0.638,0.371,0.295)>	<[s1.97,s3.05],(0.621,0.371,0.314)>
A_3	<[s2.79,s4.31],(0.661,0.371,0.294)>	<[s2.67,s4.18],(0.604,0.371,0.311)>
A_4	<[s2.25,s3.77],(0.672,0.378,0.348)>	<[s1.86,s3.53],(0.636,0.378,0.357)>

Step 2. According to the aggregating results shown in Table 2, we can calculate the expected value $EX(\tilde{p}_i)$ (i=1, 2, 3, 4) of the collective spherical uncertain linguistic value $\tilde{p}_i\,(i=1,2,3,4)$, and the expected values of the smart phones are shown in Table 3.

Table 3
Expected values of smart phones

Alt.	SULHA	SULHG	GSULWA($\lambda=1$)	GSULWG($\lambda=1$)
A_1	2.591	2.188	2.332	1.973
A_2	1.585	1.615	1.781	1.621
A_3	2.651	2.553	2.363	2.192
A_4	1.970	1.775	1.951	1.708

Step 3. According to the expected values shown in Table 3 and the ranking of the smart phones are shown in Table 4. Note that " $\succ$ " means "preferred to". As we can see, depending on the aggregation operators used, the ordering of the smart phones is the same, and the best smart phone is A_3.

Table 4
Ordering of the smart phones

Operators	Ordering
SULHA	$A_3 \succ A_1 \succ A_4 \succ A_2$
SULHG	$A_3 \succ A_1 \succ A_4 \succ A_2$
GSULWA($\lambda=1$)	$A_3 \succ A_1 \succ A_4 \succ A_2$
GSULWG($\lambda=1$)	$A_3 \succ A_1 \succ A_4 \succ A_2$

6.2 Comparative analysis and discussion

In this comparison, we used the SFNWAA operator proposed by S. Ashraf *et al.*, [63] to solve the example used in this paper. As the attribute information in [63] occurs in the form of SFNs, we then used the decision matrix from Table 1 without uncertain linguistic variables, as follows Table 5.

Table 5
The spherical fuzzy decision matrix

Alt.	C_1	C_2	C_3	C_4
A_1	<0.7,0.4,0.2>	< 0.6,0.4,0.3>	<0.6,0.5,0.4>	<0.5,0.2,0.3>
A_2	<0.7,0.5,0.3>	<0.5,0.7,0.2>	<0.7,0.3,0.3>	<0.6,0.2,0.4>
A_3	< 0.4,0.7,0.4>	<0.7,0.3,0.2>	<0.8,0.3,0.3>	<0.6,0.3,0.3>
A_4	<0.8,0.3,0.4>	<0.7,0.6,0.3>	<0.6,0.4,0.3>	<0.5,0.3,0.4>

From Table 6, we observe that although the expected values of the alternatives obtained by the proposed operators in this paper are different, the final ranking is exactly the same, all of which are $A_3 \succ A_1 \succ A_4 \succ A_2$. Meanwhile, the ranking results obtained by SFNWAA operator in [63] are not the same as those got by the proposed operators, the main reason is that the uncertain factors in decision information are not considered. But the best alternative is A_3, which shows the effectiveness of decision making by using the methods proposed in this paper.

Table 6
Ranking of different methods

Operators	Expected values of A_i	Ordering
SULHA	EX(A_1)=2.591; EX(A_2)=1.585; EX(A_3)=2.651; EX(A_4)=1.970	$A_3 \succ A_1 \succ A_4 \succ A_2$
SULHG	EX(A_1)=2.188; EX(A_2)=1.615; EX(A_3)=2.553; EX(A_4)=1.775	$A_3 \succ A_1 \succ A_4 \succ A_2$
GSULWA($\lambda = 1$)	EX(A_1)=2.332; EX(A_2)=1.781; EX(A_3)=2.363; EX(A_4)=1.951	$A_3 \succ A_1 \succ A_4 \succ A_2$
GSULWG($\lambda = 1$)	EX(A_1)=1.973; EX(A_2)=1.621; EX(A_3)=2.669; EX(A_4)=1.708	$A_3 \succ A_1 \succ A_4 \succ A_2$
SFNWAA [63]	EX(A_1)=0.655; EX(A_2)=0.657; EX(A_3)=0.665; EX(A_4)=0.648	$A_3 \succ A_2 \succ A_1 \succ A_4$

When parameter λ is removed from different values, different results can be obtained. Table 7 and Table 8 show the expected values of alternatives corresponding to different parameter λ.

Table 7
Expected values of alternatives with different λ by GSULWA operator

λ	EX(A_1)	EX(A_2)	EX(A_3)	EX(A_4)	Ranking
0.5	2.162	1.711	2.308	1.844	$A_3 \succ A_1 \succ A_4 \succ A_2$
1	2.332	1.781	2.363	1.951	$A_3 \succ A_1 \succ A_4 \succ A_2$
2	2.663	1.924	2.473	2.147	$A_1 \succ A_3 \succ A_4 \succ A_2$
3	2.948	2.061	2.580	2.302	$A_1 \succ A_3 \succ A_4 \succ A_2$
5	3.358	2.291	2.768	2.510	$A_1 \succ A_3 \succ A_4 \succ A_2$
7	3.613	2.456	2.915	2.640	$A_1 \succ A_3 \succ A_4 \succ A_2$

Table 8
Expected values of alternatives with different λ by GSULWG operator

λ	EX(A_1)	EX(A_2)	EX(A_3)	EX(A_4)	Ranking
0.5	1.981	1.627	2.206	1.714	$A_3 \succ A_1 \succ A_4 \succ A_2$
2	1.957	1.606	2.163	1.695	$A_3 \succ A_1 \succ A_4 \succ A_2$
3	1.942	1.593	2.135	1.682	$A_3 \succ A_1 \succ A_4 \succ A_2$
5	1.917	1.571	2.088	1.659	$A_3 \succ A_1 \succ A_4 \succ A_2$
7	1.899	1.553	2.052	1.642	$A_3 \succ A_1 \succ A_4 \succ A_2$

It can be observed from Table 7 that with the increase of parameter λ, the expected value of each alternative also increases, while the expected value of each alternative in Table 8 decreases. In practical decision-making, decision makers can choose different parameter λ according to their preference degree.

Figures 1 to 4 show how the expectation values and exact values of the proposed generalized aggregation operators change with the increase of the parameter λ. The parameter λ ranges from 0 to 30. For the GSULWA operator, the expectation value curves of A_1 and A_3 intersect at point (1.0693，2.3775) in Figure 1. According to Theorem 2, it can be seen from Figure 2 that A_3 is preferred over A_1. The ranking of each alternative in this process is relatively stable with respect to the parameter λ. For the GSULWG operator, within the parameter λ range of 0 to 30, the expectation values and exact values of the operator first decrease and then suddenly increase. The priority order of each alternative in this process is quite stable with respect to the parameter λ.

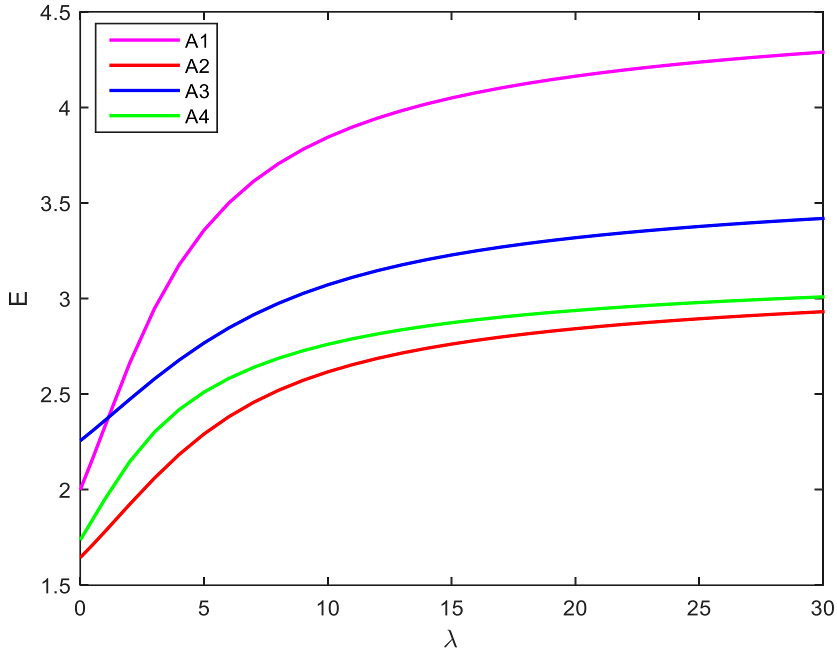

Fig. 1. GSULWA operator's expectation value with parameter λ

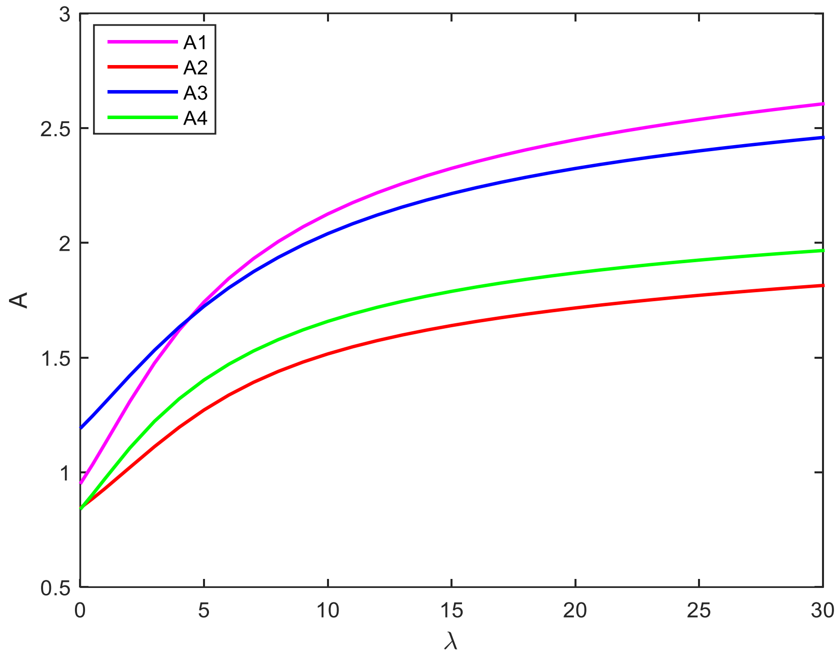

Fig. 2. GSULWA operator's accuracy value with parameter λ

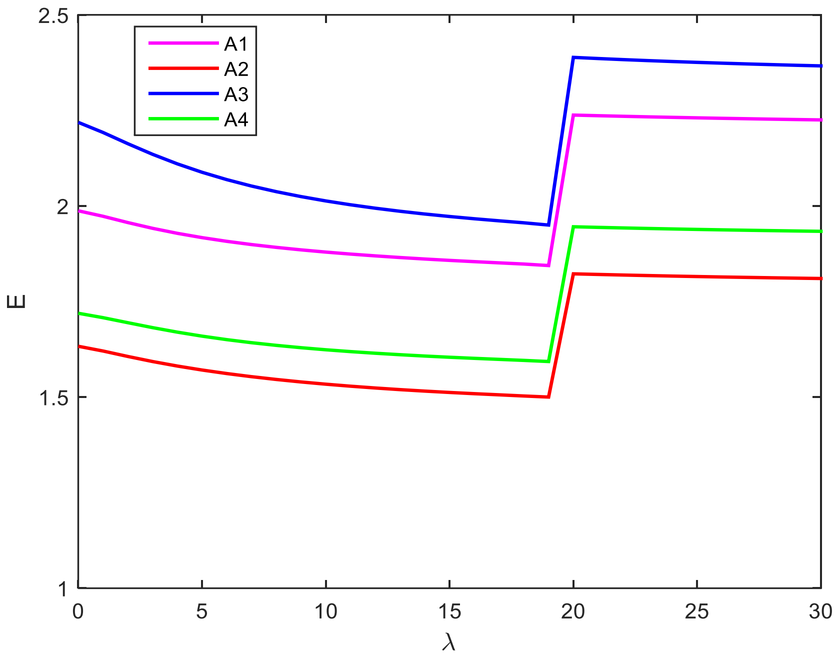

Fig. 3. GSULWG operator's expected value with parameter λ

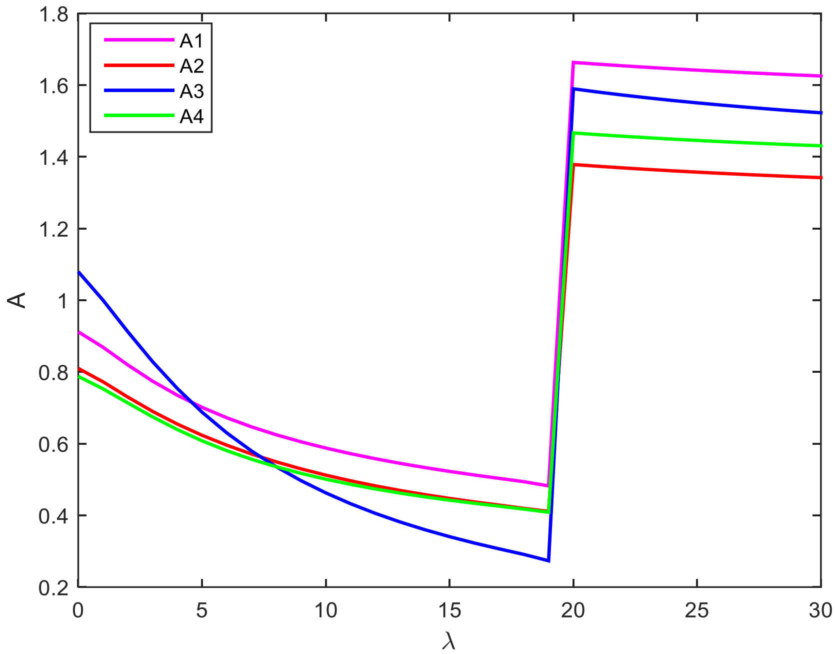

Fig. 4. GSULWG operator's accuracy value with parameter λ

7. Conclusion

Spherical fuzzy sets (SFSs) are a new extension of Cuong's picture fuzzy sets (PFSs), spherical uncertain linguistic sets [SULSs] are also a extension of picture uncertain linguistic sets (PULSs) proposed by Wu [64]. In this paper, we explore the multiple attribute decision making problems with spherical uncertain linguistic information. Then, some spherical uncertain linguistic aggregation operators such as spherical uncertain linguistic weighted average (SULWA) operator, spherical uncertain linguistic ordered weighted average (SULOWA) operator, spherical uncertain linguistic hybrid average (SULHA) operator, generalized spherical uncertain linguistic weighted average

(GSULWA) operator, spherical uncertain linguistic weighted geometric (SULWG) operator, spherical uncertain linguistic ordered weighted geometric (SULOWG) operator, spherical uncertain linguistic hybrid geometric (SULHG) operator, generalized spherical uncertain linguistic weighted geometric (GSULWG) operator are put forward by utilizing arithmetic and geometric operation. The prominent characteristic of these proposed operators are studied. Then, we have used SULHA, SULHG, GSULWA and GSULWG operators to solve the multiple attribute decision making problems under spherical uncertain linguistic environment. Finally, a practical example for smart phones evaluation of shipping website is given to verify the presented approach and to demonstrate its practicality and effectiveness. In the future, we will continue working on the extension and application of the developed operators to other domains.

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Conflicts of Interest
The authors declare no conflicts of interest.

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